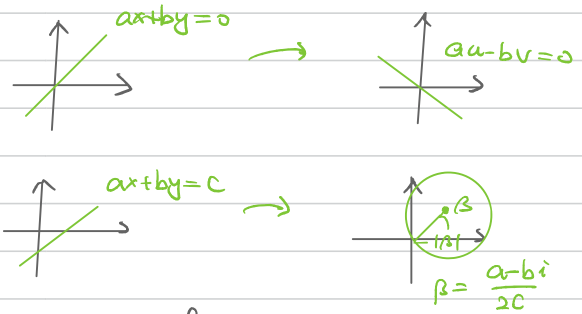
pf

② If $S$ is a line, then $\exists\, a,b,c \in \mathbb{R}$ s.t.

(✱) $z = x+iy \in S \Rightarrow ax+by = c$

Let $\alpha = a - bi \Rightarrow \mathrm{Re}(\alpha z) = ax+by$

(✱): $\mathrm{Re}(\alpha z) = c$ or $\alpha z + \overline{\alpha z} = 2c$




case1: $c=0$ i.e. $0 \in S$

If $w = u+iv$, then $\alpha z = \frac{\alpha(u-iv)}{u^2+v^2} \Rightarrow$ (✱): $au - bv = 0$ —— line

case2: $c \neq 0$

(✱): $w\bar{w} - \frac{\alpha}{2c}\bar{w} - \frac{\bar{\alpha}}{2c} w = 0 \Rightarrow w\bar{w} - \beta\bar{w} - \bar{\beta}w + |\beta|^2 = |\beta|^2$, where $\beta = \frac{\alpha}{2c}$

$\Rightarrow |w-\beta|^2 = |\beta|^2$ —— circle #

Thm 13.11

$$f(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0$$

maps circles and lines onto circles and lines

pf

If $c=0$, then $f$ is linear and the result is immediate.

If $c \neq 0$,

$$f(z) = \frac{az+b}{cz+d} = \frac{1}{c}\left(a - \frac{ad-bc}{cz+d}\right) = (f_3 \circ f_2 \circ f_1)(z)$$

where

$f_1(z) = cz+d$, $\underbrace{f_2(z) = \frac{1}{z}}_{\text{Lemma 13.10}}$, $f_3(z) = \frac{a}{c} - \frac{ad-bc}{c} z$

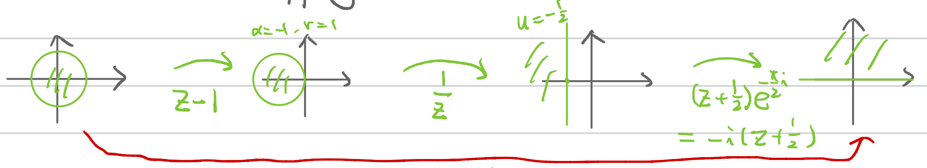
linear

$\Rightarrow$ the result follows. #

Example1

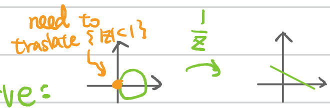
Find a conformal mapping $f: \{|z|<1\} \twoheadrightarrow \{\mathrm{Im}\, z > 0\}$

sol




$$-i\left(\frac{1}{z-1} + \frac{1}{2}\right) = -i\left(\frac{\frac{1}{2}z+\frac{1}{2}}{z-1}\right) = -\frac{i}{2}\left(\frac{z+1}{z-1}\right) \quad \#$$

Observe:



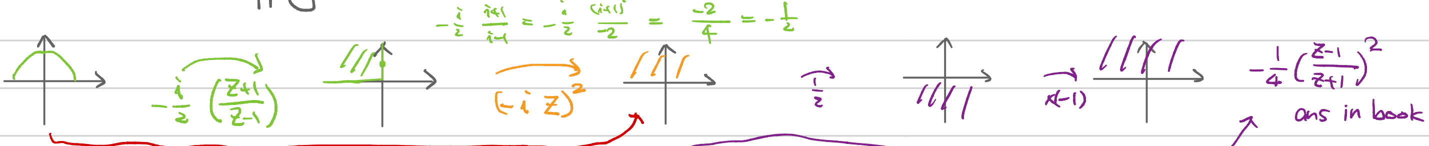

Note: $-i\left(\frac{z+1}{z-1}\right)$ is also ok.

Example2 (p.180)

Find a conformal mapping $f: \{|z|<1, \mathrm{Im}\, z>0\} \longrightarrow \{\mathrm{Im}\, z > 0\}$

sol

$-\frac{i}{2}\frac{i+1}{i-1} = -\frac{i}{2}\frac{(i+1)^2}{-2} = \frac{-2}{4} = -\frac{1}{2}$

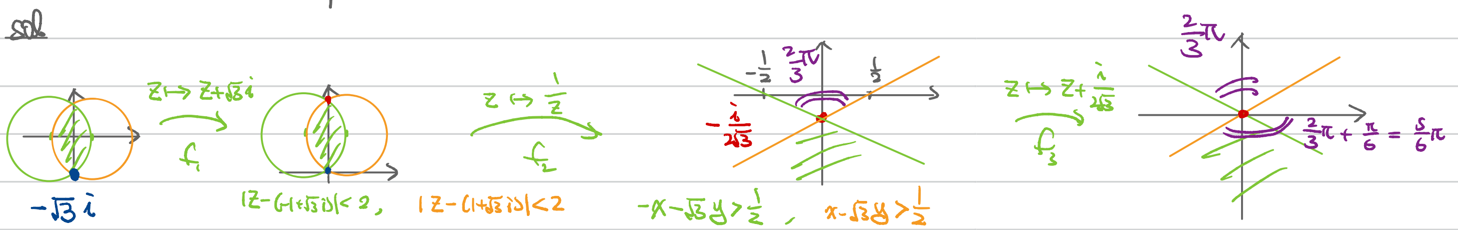



$$\left(\frac{-i}{2}\left(\frac{z+1}{z-1}\right)\right)^2 = \frac{1}{4}\frac{(z+1)^2}{(z-1)^2} \quad \#$$

Example3

Find a conformal map $f: \{|z-1|<2, |z+1|<2\} \twoheadrightarrow \{|z|<1\}$

sol




$|z-(-1+\sqrt{3}i)|<2$, $|z-(1+\sqrt{3}i)|<2$

$-x-\sqrt{3}y > \frac{1}{2}$, $x - \sqrt{3}y > \frac{1}{2}$

$-x-\sqrt{3}y > 0$, $x-\sqrt{3}y > 0$

...

sol

$\times e^{i\frac{5}{6}\pi}$ $f_4$ $\frac{2}{3}\pi$ (here $\sqrt{i} = e^{\frac{\pi}{4}i}$) $z \mapsto z^{\frac{3}{2}}$ $f_5$ $f_6$ $z \mapsto \frac{z-i}{z+i}$

by Example 1, we can put the inverse of $-i\left(\frac{z+1}{z-1}\right) = \frac{-iz-i}{z-1}$ — $\frac{-z+i}{-z-i} = \frac{z-i}{z+i}$

$\{|z|<1\}$

ans $= f_6 \circ f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$ #

Def 13.12

A conformal mapping of a region onto itself is called an automorphism of the region

Remark (13.13 – 13.14)

① If $f: D_1 \twoheadrightarrow D_2$ is conformal, then

a. any conformal mapping $D_1 \twoheadrightarrow D_2$ is of the form $g \circ f$

b. any automorphism $D_1 \twoheadrightarrow D_1$ is of the form $f^{-1} \circ g \circ f$

where $g: D_2 \twoheadrightarrow D_2$ is an automorphism

② The only automorphisms $f: \{|z|<1\} \to \{|z|<1\}$ with $f(0)=0$ are $f(z) = e^{i\theta} z$

by Schwarz Lemma. See the proof of uniqueness for Riemann Mapping Thm

Thm (Thm 13.15, 13.16, 13.17)

Let $U = \{|z|<1\}$, $H^+ = \{\operatorname{Im} z > 0\}$,

① If $f: U \twoheadrightarrow U$ is an automorphism, then

$f(z) = e^{i\theta}\left(\frac{z-\alpha}{1-\bar{\alpha}z}\right)$, $\exists\, |\alpha|<1$, $\theta \in \mathbb{R}$ ← recall: $B_\alpha: U \to U$, $B_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$ p95

② If $g: H^+ \twoheadrightarrow U$ is a conformal mapping, then

$g(z) = e^{i\theta}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right)$, $\exists\, \operatorname{Im}\alpha > 0$, $\theta \in \mathbb{R}$ ← see Example 1: $-i\left(\frac{z+1}{z-1}\right) = \frac{-iz-i}{z-1}$ has inverse $\frac{-z+i}{-z-i} = \frac{z-i}{z+i}$

③ If $h: H^+ \twoheadrightarrow H^+$ is an automorphism, then

$h(z) = \frac{az+b}{cz+d}$

for some $a,b,c,d \in \mathbb{R}$, $ad-bc>0$

pf

① Recall: $B_\alpha: U \to U$, $B_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$, $|\alpha|<1$, is an automorphism s.t. $B_\alpha(\alpha)=0$. If $f: U \twoheadrightarrow U$ is an auto, then $B_{f(0)} \circ f$ is an automorphism mapping $0 \mapsto 0$. By Remark ②, $B_{f(0)} \circ f(z) = e^{i\theta} z \Rightarrow f(z) = B^{-1}_{f(0)}(e^{i\theta} z) = \frac{e^{i\theta}z + f(0)}{1+\overline{f(0)}e^{i\theta}z} = e^{i\theta}\left(\frac{z + f(0)e^{-i\theta}}{1+\left(\overline{f(0)e^{-i\theta}}\right)z}\right)$ ∎

② Step 1 Check $g_0(z) = \frac{z-i}{z+i}$ is a conformal map $H^+ \twoheadrightarrow U$. Step 2

$$\begin{tikzcd} H^+ \arrow[r, two heads, "\text{any auto}", "g"'] \arrow[dr, "g_0"'] & U \\ & U \arrow[u, "g \circ g_0^{-1}(z) = f(z) \text{ in ①}"'] \end{tikzcd}$$

$\Rightarrow g = f \circ g_0$ then compute ∎

③
$$\begin{tikzcd} H^+ \arrow[r, "\text{any auto}", "h"'] \arrow[d, "g_0"'] & H^+ \arrow[d, "g_0"] \\ U \arrow[r, "f"'] & U \end{tikzcd}$$
$\Rightarrow h = g_0^{-1} \circ f \circ g_0$ ∎

See p.183–184 for more details. #

Thm 13.23

The unique bilinear transformation $w = f(z)$ mapping $z_1, z_2, z_3$ to $w_1, w_2, w_3$, respectively, is given by

$$\frac{(w-w_2)(w_3-w_1)}{(w-w_1)(w_3-w_2)} = \frac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)}$$

pf: skip

Example

Find a bilinear transformation mapping

$0 \mapsto i$, $-1 \mapsto 0$, $i \mapsto -1$

sol

By Thm 13.23,

$$\frac{(w-0)(-1-i)}{(w-i)(-1-0)} = \frac{(z-(-1))(i-0)}{(z-0)(i+1)}$$

$$= \frac{w(1+i)}{w-i} = \frac{i}{i+1}\frac{(z+1)}{z}$$

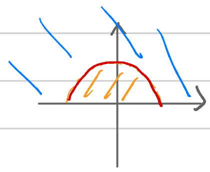
$\Leftrightarrow wz(1+i)^2 = i(w-i)(z+1) = i(z+1)w + (z+1)$ $\quad [(1+i)^2 = 2i]$

$w = \frac{z+1}{iz-i} = -i \cdot \frac{z+1}{z-1}$ #

Problem 10 in HW6:

Suppose $f$ is analytic in the semi-disc: $|z|<1$, $\operatorname{Im} z>0$, continuous on $|z|\le 1$, $\operatorname{Im} z>0$, and real on the semi-circle $|z|=1$, $\operatorname{Im} z>0$. Show that if we set

$$g(z)=\begin{cases} f(z), & |z|\le 1,\ \operatorname{Im} z>0\\ \overline{f(1/\bar z)}, & |z|>1,\ \operatorname{Im} z>0,\end{cases}$$



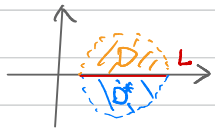
then $g$ is analytic in $\{\operatorname{Im} z>0\}$

pf

Recall (Schwarz reflection principle Thm 7.8)

Suppose $f$ is analytic in a domain $D$ and continuous in $\bar D$. Suppose $\partial D$ contains a line segment $L$ on the real axis, and $f(L)\subseteq \mathbb{R}$. Then

$$g(z):=\begin{cases} f(z), & z\in D\cup L\\ \overline{f(\bar z)}, & z\in D^*\end{cases}$$



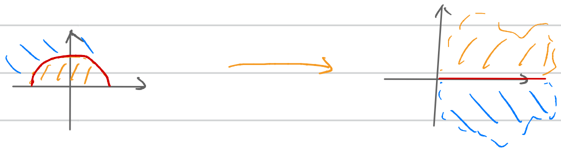

is analytic in $D\cup L\cup D^*$, where $D^*=\{z\in\mathbb{C}: \bar z\in D\}$

Idea: transfer Schwarz reflection principle by a suitable conformal map

Need: a conformal map which maps the semi-circle $|z|=1$, $\operatorname{Im} z>0$ to $L$ on the real axis.

Construction of conformal map

Recall in the proof of Lemma 3.10, we computed:



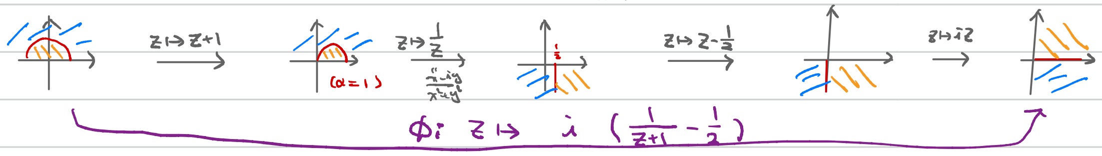
need → (i) $|z-\alpha|=|\alpha|$ $\xrightarrow{\frac{1}{z}}$ $x_0u-y_0v=\frac{1}{2}$

z/xy-plane w/uv-plane $\alpha=x_0+iy_0$

(ii) $|z-\alpha|=r$, $r\ne|\alpha|$ $\xrightarrow{\frac{1}{z}}$ $|w-\beta|=\left|\frac{r}{|\alpha|^2-r^2}\right|$

$\beta=\frac{\bar\alpha}{|\alpha|^2-r^2}$

So




Transfer the principle

Note if $w=i\left(\frac{1}{z+1}-\frac{1}{2}\right)$, then $\frac{w}{i}+\frac{1}{2}=\frac{1}{z+1}\Rightarrow (z+1)(2w+i)=2i\Rightarrow z=(2i-2w-i)/(2w+i)$

Let $\phi(z):=i\left(\frac{1}{z+1}-\frac{1}{2}\right)=\frac{i-iz}{2z+2}\leftarrow\begin{pmatrix}-i & i\\ 2 & 2\end{pmatrix}$ $\psi(w)=\frac{i-2w}{2w+i}\leftarrow\begin{pmatrix}-2 & i\\ 2 & i\end{pmatrix}\sim\begin{pmatrix}2 & -i\\ -2 & -i\end{pmatrix}\cdot\frac{1}{\det}$

Then $f\circ\psi$ is analytic in $D\{\operatorname{Im} w>0, \operatorname{Re} w>0\}$, continuous in $\bar D\{\operatorname{Im} w\ge 0, \operatorname{Re} w>0\}$, real on $L\{\operatorname{Im} w=0, \operatorname{Re} w>0\}$

By Shwarz reflection principle,

$$h(w):=\begin{cases} f(\psi(w)) & \text{if } \operatorname{Im} w\ge 0,\ \operatorname{Re} w>0\\ \overline{f(\psi(\bar w))} & \text{if } \operatorname{Im} w<0,\ \operatorname{Re} w>0\end{cases}$$

is analytic in $\{\operatorname{Re} w>0\}$

$\Rightarrow g:=h\circ\phi$ is analytic in $\{\operatorname{Im} z>0\}$.

The formula

$$g(z)=\begin{cases} f(\psi(\phi(z)))=f(z) & \text{if } |z|\le 1,\ \operatorname{Im} z>0\\ \overline{f(\psi(\overline{\phi(z)}))} & \text{if } |z|>1,\ \operatorname{Im} z>0\end{cases}$$

and

$$\psi(\overline{\phi(z)})=\psi\left(\frac{-i+i\bar z}{2\bar z+2}\right)=\left(i-2\left(\frac{-i+i\bar z}{2\bar z+2}\right)\right)\Big/\left(2\left(\frac{-i+i\bar z}{2\bar z+2}\right)+i\right)=\left(i-\frac{-i+i\bar z}{\bar z+1}\right)\Big/\left(\frac{-i+i\bar z}{\bar z+1}+i\right)$$

$$=\frac{\cancel{i\bar z}+i+i-\cancel{i\bar z}}{\cancel{-i}+i\bar z+i\bar z+\cancel{i}}=\frac{2i}{2i\bar z}=\frac{1}{\bar z}$$

So

$$g(z)=\begin{cases} f(z) & \text{if } |z|\le 1,\ \operatorname{Im} z>0\\ \overline{f(1/\bar z)} & |z|>1,\ \operatorname{Im} z>0\end{cases}$$

#

~~Exer~~ Try to get other forms of this principle.