Example2

Let $f(z)=z^2$. $\Rightarrow f'(z)=2z\neq 0 \ \forall z\neq 0$ $\Rightarrow$ f is conformal, locally 1-1 (NOT 1-1) throughout $\mathbb{C}-\{0\}$ see p. 172 for picture

Def 13.5

Let $k\in\mathbb{N}$. We say f is a k-to-1 mapping of D_1 onto D_2 if $\forall \alpha\in D_2$, the eq $f(z)=\alpha$ has k roots (counting multiplicity) in D_1.

Example

Let $f(z)=z^k$, $k\in\mathbb{N}$, $\delta>0$. Then $f: D(0;\delta)\longrightarrow D(0;\delta^k)$ is k-to-1.

Thm 13.7 (cf. Thm 13.4.)

Suppose f is analytic at z_0 with $f'(z_0)=0$.
If f is not constant in a nbd of z_0, then f is a k-to-1 mapping and f magnifies angles at z_0 by k in a nbd of z_0, where k is the least positive integer for which $f^{(k)}(z_0)\neq 0$

pf

① Without loss of generality, assume $f(z_0)=0$.
By hypothesis, the Taylor expansion of f about z_0 is of the form

$$f(z)= (z-z_0)^k \underbrace{\left(a_k + a_{k+1}(z-z_0) + a_{k+2}(z-z_0)^2 + \cdots\right)}_{=:\, g(z)}$$

with $a_k = f^{(k)}(z_0)/k! \neq 0$

② Since $g(z_0)\neq 0$, $\exists\,\delta>0$ s.t. $g(z)\neq 0 \ \forall z\in D(z_0;\delta)$ ← simply connected

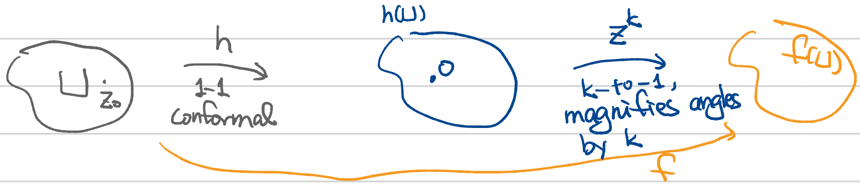
$\Rightarrow \exists\, g^{1/k}(z)$ analytic in $D(z_0;\delta)$ s.t. $(g^{1/k}(z))^k = g(z)$ $\forall z\in D(z_0;\delta)$

Let

$$h(z)= (z-z_0)\, g^{1/k}(z) \quad \leftarrow \text{analytic}$$

$\Rightarrow f(z)=(h(z))^k$, $h(z_0)=0$, $h'(z_0)=g^{1/k}(z_0)\neq 0$.

$\Rightarrow \exists$ nbd U of z_0 in $D(z_0;\delta)$ s.t. h is 1-1, conformal in U



#

Thm 13.8

Suppose f is a 1-1 analytic function in a region D. Then
a. f^{-1} exists and is analytic in $f(D)$
b. f and f^{-1} are conformal in D and $f(D)$, respectively.

pf

f is 1-1 $\overset{\text{Thm 13.7}}{\Longrightarrow}$ $f'\neq 0$ $\overset{\text{Prop 3.5}}{\Longrightarrow}$ f^{-1} is also analytic and $(f^{-1})' = \frac{1}{f'} \neq 0$

$\Rightarrow f$ and f^{-1} are both conformal. #

Def 13.9

a. A 1-1 analytic mapping is called a conformal mapping
b. Two regions D_1 and D_2 are conformally equivalent if $\exists$ conformal mapping $D_1 \xrightarrow[\text{onto}]{1-1} D_2$
Such a bijective conformal mapping is called a conformal equivalence or biholomorphism

Riemann Mapping Theorem (see §14.2)

Any two simply connected domains (except $\mathbb{C}$) are conformally equivalent

Note that if we have a conformal equivalence from any proper s-cnt domain (i.e. s-cnt domain, $\neq \mathbb{C}$) to unit disc ($\mathbb{U} = \{|z|<1\}$)

$$\text{any s-cnt domain} \neq \mathbb{C} \xrightarrow[\text{analytic}]{\text{bijective}} \mathbb{U}$$

then we can get a conformal equivalence between any two proper s-cnt domains, R_1, R_2, by composition

$$\begin{tikzcd} R_1 \arrow[r, "f_1"] \arrow[rr, bend right, "f_2^{-1}\circ f_1"'] & \mathbb{U} & R_2 \arrow[l, "f_2"'] \end{tikzcd}$$

In fact, one has the following (better) formulation of Riemann Mapping Thm:

Riemann Mapping Thm (p. 200)

For any simply connected domain R which is not $\mathbb{C}$ and any $z_0 \in R$, there exists a unique conformal equivalence $\varphi: R \to \mathbb{U}$ s.t.

$$\varphi(z_0) = 0 \quad \text{and} \quad \varphi'(z_0) > 0$$

Remark

① For $\alpha \in \mathbb{U}$, we considered the function

$$B_\alpha : \mathbb{U} \longrightarrow \mathbb{U}, \quad B_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$$

↙ p. 95, Ch7

which is $\overset{(i)}{\text{analytic}}$, $\overset{(ii)}{B_\alpha(\alpha) = 0}$,

(iii) $B_\alpha'(\alpha) = \left.\dfrac{1-\bar{\alpha}z - (z-\alpha)(-\bar{\alpha})}{(1-\bar{\alpha}z)^2}\right|_{z=\alpha} = \dfrac{1-|\alpha|^2}{(1-|\alpha|^2)^2} = \dfrac{1}{1-|\alpha|^2} > 0$

(iv) $B_\alpha(z_1) = B_\alpha(z_2) \Rightarrow (z_1-\alpha)(1-\bar{\alpha}z_2) = (z_2-\alpha)(1-\bar{\alpha}z_1) = z_1 - \cancel{\bar{\alpha}z_1z_2} - \cancel{\alpha} + |\alpha|^2 z_2 = z_2 - \cancel{\bar{\alpha}z_1z_2} - \cancel{\alpha} + |\alpha|^2 z_1$

$\Rightarrow (z_1 - z_2)(1-|\alpha|^2) = 0 \Rightarrow z_1 = z_2 \Rightarrow$ 1-1

enough for 1-1, onto

(v) $\forall \omega \in \mathbb{U}$, $B_\alpha(z) = \omega \Rightarrow z - \alpha = \omega - \bar{\alpha}z\omega \Rightarrow z = \dfrac{\omega+\alpha}{1+\bar{\alpha}\omega} = B_{-\alpha}(\omega) \Rightarrow B_\alpha$ is onto, $\boxed{B_\alpha^{-1} = B_{-\alpha}}$

So $B_\alpha : \mathbb{U} \to \mathbb{U}$ is the unique conformal equivalence s.t. $B_\alpha(\alpha) = 0$, $B_\alpha'(\alpha) > 0$.

② Problem 10 in HW6: use the conformal maps

to transfer Schwartz reflection principle (Thm 7.8)

⇝ See Thm 13.11

Recall (Schwarz Lemma, Lemma 7.2)

Suppose $f: \mathbb{U} \to \mathbb{U}$ is analytic, $f(0) = 0$. Then

(i) $|f(z)| \leq |z| \quad \forall z \in \mathbb{U}$

(ii) $|f'(0)| \leq 1$

And "=" holds in either (i) or (ii) $\Leftrightarrow f(z) = e^{i\theta}z$

③ The proof of uniqueness is easy:

Suppose $\varphi_1, \varphi_2 : R \to \mathbb{U}$ satisfy the properties in thm.

$\Rightarrow \Phi = \varphi_1 \circ \varphi_2^{-1} : \mathbb{U} \to \mathbb{U}$ is analytic and $\Phi(0) = 0$

By Schwarz's Lemma (Lemma 7.2), $|\Phi(z)| \leq |z|$

Similarly, $|\Phi^{-1}(z)| \leq |z|$

$\Rightarrow |\Phi(z)| \leq |z| = |\Phi^{-1}\Phi(z)| \leq |\Phi(z)|$

$\Rightarrow |z| = |\Phi(z)|$

$\Rightarrow$ by Schwarz's Lemma again, $\Phi(z) = e^{i\theta}z$ for some θ.

By assumption, $\Phi'(0) = e^{i\theta} > 0 \Rightarrow e^{i\theta} = 1 \Rightarrow \Phi(z) = z \Rightarrow \varphi_1 = \varphi_2$ #

④ The proof of existence is difficult:

See p. 201 – p. 204. We skip it here

Instead, we will show explicit conformal maps between various s-cnt domains.

exer

Use Riemann Mapping Thm to show $\mathbb{C}$ is NOT conformally equivalent to a s-cnt domain $R \neq \mathbb{C}$

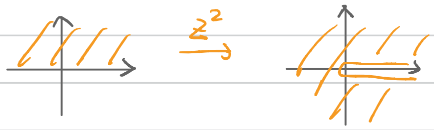
(exer 10, Ch 14)

Explicit conformal maps

I. Elementary transformations:

(i) $\omega = az+b$, $a \neq 0$, is the composition of 3 maps

1. $\omega_1 = kz$, $k = |a|$ —— magnification
2. $\omega_2 = e^{i\theta} z$, $\theta = \operatorname{Arg} a$ —— rotation
3. $\omega_3 = z+b$ —— translation

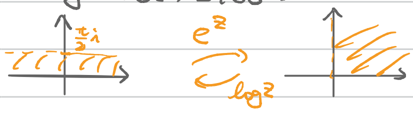


(ii) $\omega = z^\alpha$, $\alpha > 0$, given by $e^{\alpha \log z}$ is analytic in a simply connected domain $\not\ni 0$

if $z = re^{i\theta}$, $\omega = r^\alpha e^{i\alpha\theta}$ which is conformal in $\{\theta_1 < \operatorname{Arg} z < \theta_2,\ z \neq 0\}$, $\theta_2 - \theta_1 \leq \frac{2\pi}{\alpha}$

(iii) $\omega = e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$

is conformal in $\{y_1 < y < y_2\}$, $y_2 - y_1 \leq 2\pi$



II. Bilinear transformations (a.k.a. Möbius transformations):

The map given by
$$f(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0$$
is called a bilinear transformation / Möbius transformation

Remark

① $\frac{az+b}{cz+d} = \omega \Rightarrow z = \frac{d\omega - b}{-c\omega + a}$ $\quad \forall \omega \neq \frac{a}{c}$

So $f: \mathbb{C} - \{-\frac{d}{c}\} \xrightarrow[\text{onto}]{1-1} \mathbb{C} - \{\frac{a}{c}\}$ is conformal

② $f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0 \quad \forall z \neq -\frac{d}{c}$

③ $B_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$ (p.95) is an example

④ If $g = \frac{a'z+b'}{c'z+d'}$, then

$f \circ g = \left(a \frac{a'z+b'}{c'z+d'} + b\right) \Big/ \left(c \frac{a'z+b'}{c'z+d'} + d\right) = \frac{(aa'+bc')z + (ab'+bd')}{c'z+d'} \cdot \frac{c'z+d'}{(ca'+dc')z + (cb'+dd')}$

$= \frac{(aa'+bc')z + (ab'+bd')}{(ca'+dc')z + (cb'+dd')}$

Note: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{pmatrix}$

det $\neq 0$ $\quad \Rightarrow \quad$ det $\neq 0$

So $\left\{ \frac{az+b}{cz+d},\ ad-bc \neq 0 \right\}$ is a group under composition. ($\cong GL_2(\mathbb{C}) / \mathbb{C}^* I_2$)

Lemma 13.10

If S is a circle or a line, and $f(z) = \frac{1}{z}$, then $f(S)$ is also a circle or a line

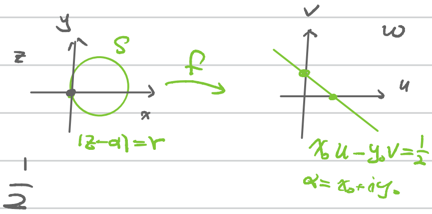
pf (See exer 27, 28 in Ch1 for a different proof.)

① Let $S = C(\alpha; r) = \{\alpha + re^{i\theta} : \theta \in [0, 2\pi]\}$, "$f(S)$" $= \{\omega = \frac{1}{z} : z \in S,\ z \neq 0\}$

Note: $C(\alpha; r) = \{|z-\alpha|^2 = r^2 = (z-\alpha)(\bar{z}-\bar{\alpha})\}$

eq $\Leftrightarrow$ $z\bar{z} - \alpha\bar{z} - \bar{\alpha}z = r^2 - |\alpha|^2$

or ⊛ $\frac{1}{\omega\bar{\omega}} - \frac{\alpha}{\bar{\omega}} - \frac{\bar{\alpha}}{\omega} = r^2 - |\alpha|^2$ $\quad (z = \frac{1}{\omega})$



case 1 $r = |\alpha|$, i.e. $0 \in S$

⊛: $1 - \alpha\omega - \bar{\alpha}\bar{\omega} = 1 - 2\operatorname{Re}(\alpha\omega) = 0 \Leftrightarrow \operatorname{Re}(\alpha\omega) = \frac{1}{2}$

If $\alpha = x_0 + iy_0$, $\omega = u + iv$, then $\operatorname{Re}(\alpha\omega) = ux_0 - vy_0$

$\Rightarrow f(S) = \{\omega = u + iv \in \mathbb{C} : x_0 u - y_0 v = \frac{1}{2}\}$ is a line

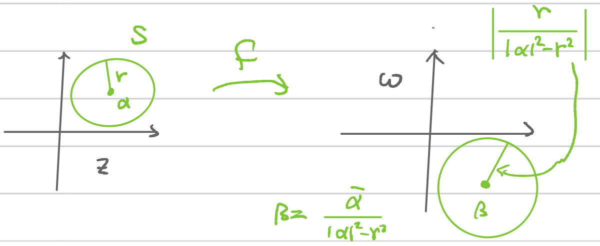
case 2 $r \neq |\alpha|$, i.e. $0 \notin S$

⊛: $\omega\bar{\omega} - \left(\frac{\bar{\alpha}}{|\alpha|^2 - r^2}\right)\bar{\omega} - \left(\frac{\alpha}{|\alpha|^2 - r^2}\right)\omega = \frac{-1}{|\alpha|^2 - r^2}$,

and setting
$$\beta = \frac{\bar{\alpha}}{|\alpha|^2 - r^2}$$

⊛ $\Leftrightarrow$ $\omega\bar{\omega} - \beta\bar{\omega} - \bar{\beta}\omega + |\beta|^2 = \frac{r^2}{(|\alpha|^2 - r^2)^2}$

i.e. $|\omega - \beta|^2 = \left(\frac{r}{|\alpha|^2 - r^2}\right)^2$



pf

② If S is a line, then $\exists\ a, b, c \in \mathbb{R}$ s.t.

(※) $z = x+iy \in S \Rightarrow ax+by = c$

Let $\alpha = a - bi \Rightarrow \mathrm{Re}(\alpha z) = ax+by$

(※): $\mathrm{Re}(\alpha z) = c$ or $\alpha z + \overline{\alpha z} = 2c$

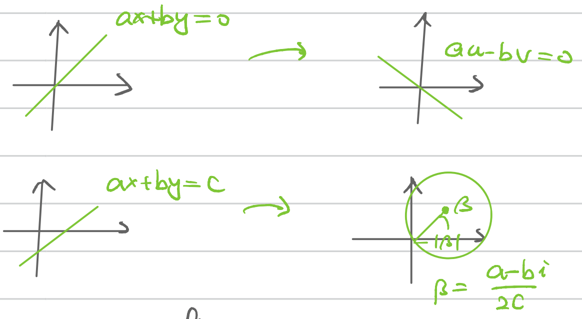
case1: $c=0$ i.e. $0 \in S$

If $w = u+iv$, then $\alpha z = \frac{\alpha(u-iv)}{u^2+v^2} \Rightarrow$ (※): $au - bv = 0$ —— line

case2: $c \neq 0$

(※): $w\bar{w} - \frac{\alpha}{2c}\bar{w} - \frac{\bar{\alpha}}{2c}w = 0 \Rightarrow w\bar{w} - \beta\bar{w} - \bar{\beta}w + |\beta|^2 = |\beta|^2$, where $\beta = \frac{\alpha}{2c}$

$\Rightarrow |w-\beta|^2 = |\beta|^2$ —— circle #



Thm 13.11

$$f(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0$$

maps circles and lines onto circles and lines

pf

If $c=0$, then f is linear and the result is immediate.

If $c \neq 0$,

$$f(z) = \frac{az+b}{cz+d} = \frac{1}{c}\left(a - \frac{ad-bc}{cz+d}\right) = (f_3 \circ f_2 \circ f_1)(z)$$

where

linear

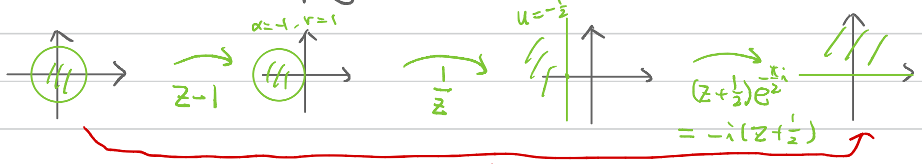
$f_1(z) = cz+d$, $\underline{f_2(z) = \frac{1}{z}}$, $f_3(z) = \frac{a}{c} - \frac{ad-bc}{c}z$ Lemma 13.10

$\Rightarrow$ the result follows. #

Example

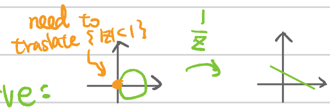
Find a conformal mapping $f: \{|z|<1\} \twoheadrightarrow \{\mathrm{Im}z > 0\}$

sol



$$-i\left(\frac{1}{z-1} + \frac{1}{2}\right) = -i\left(\frac{\frac{1}{2}z+\frac{1}{2}}{z-1}\right) = -\frac{i}{2}\left(\frac{z+1}{z-1}\right) \quad \#$$

Observe:



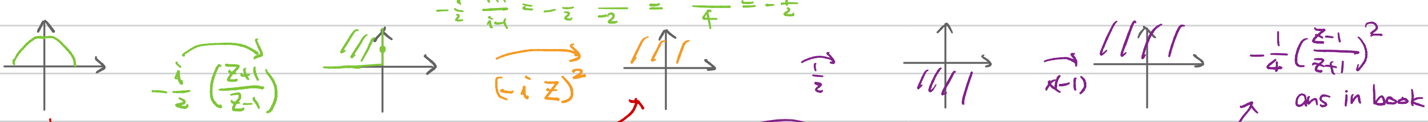
Note: $-i\left(\frac{z+1}{z-1}\right)$ is also ok.

Example (p.180)

Find a conformal mapping $f: \{|z|<1, \mathrm{Im}z>0\} \longrightarrow \{\mathrm{Im}z>0\}$

sol

$-\frac{i}{2}\frac{i+1}{i-1} = -\frac{i}{2}\frac{(i+1)^2}{-2} = \frac{-2}{4} = -\frac{1}{2}$



$$\left(-\frac{i}{2}\left(\frac{z+1}{z-1}\right)\right)^2 = \frac{1}{4}\frac{(z+1)^2}{(z-1)^2} \quad \#$$

Def 13.12

A conformal mapping of a region onto itself is called an automorphism of the region

Remark (13.13 – 13.14)

① If $f: D_1 \twoheadrightarrow D_2$ is conformal, then

a. any conformal mapping $D_1 \twoheadrightarrow D_2$ is of the form $g \circ f$

b. any automorphism $D_1 \twoheadrightarrow D_1$ is of the form $f^{-1} \circ g \circ f$

where $g: D_2 \twoheadrightarrow D_2$ is an automorphism

② The only automorphisms $f: \{|z|<1\} \rightarrow \{|z|<1\}$ with $f(0) = 0$ are $f(z) = e^{i\theta}z$

by Schwarz Lemma. See the proof of uniqueness for Riemann Mapping Thm