IV $\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$

Example

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = ?$$

sol

Note that $\int_{|z|=1} f(z)\,dz = \int_0^{2\pi} f(e^{i\theta})\, ie^{i\theta}\, d\theta$

And

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_0^{2\pi} \frac{1}{2+(e^{i\theta}+e^{-i\theta})/2}\, d\theta = \int_0^{2\pi} \frac{2e^{i\theta}\cdot i}{(e^{i\theta})^2+4e^{i\theta}+1}\, \frac{d\theta}{i}$$

$$= \left(\int_{|z|=1} \frac{dz}{z^2+4z+1}\right)\cdot \frac{2}{i}$$

$\leftarrow$ $z^2+4z+1=0 \Rightarrow z = \frac{-4\pm\sqrt{12}}{2} = -2\pm\sqrt{3}$

$|-2+\sqrt{3}|<1$, $|-2-\sqrt{3}|>1$

$$= \frac{2}{i}\cdot 2\pi i \,\mathrm{Res}\left(\frac{1}{z^2+4z+1}; -2+\sqrt{3}\right)$$

$$= 4\pi \left.\frac{1}{2z+4}\right|_{z=\sqrt{3}-2} = 4\pi \frac{1}{2\sqrt{3}-4+4} = \frac{2}{3}\sqrt{3}\,\pi \qquad \#$$

Key point:

(i) Write $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$ (ii) change to the form $\int_{|z|=1} f(z)\,dz$

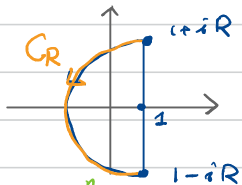
(iii) use Residue Thm

Computation of complex line integral

Example 1 (p.161-162)

Let I be the line $z(t) = 1+it$, $-\infty<t<\infty$.

$$\int_I \frac{e^z}{(z+2)^3}\,dz = ?$$



sol

Let C_R be the left semicircle of radius $R>3$ centered at $z=1$.

Then $\int_{1+iR}^{1-iR} \frac{e^z}{(z+2)^3}\,dz + \int_{C_R} \frac{e^z}{(z+2)^3}\,dz = 2\pi i\,\mathrm{Res}\left(\frac{e^z}{(z+2)^3}; -2\right) = 2\pi i\, \frac{1}{2!} e^{-2} = \frac{\pi i}{e^2}$

$\leftarrow$ $e^{-2}e^{z+2} = e^{-2}\sum \frac{(z+2)^n}{n!}$

Since $|e^z| = e^{\mathrm{Re}\, z} \le e$ $\forall z\in C_R$, $\exists$ constant A s.t.

$$\left|\int_{C_R} \frac{e^z}{(z+2)^3}\,dz\right| \le \pi R\cdot \frac{A}{R^3} \longrightarrow 0 \quad \text{as } R\to\infty$$

So $\int_I \frac{e^z}{(z+2)^3}\,dz = \pi i/e^2$ #

Example 2 (p162-163)

$$\int_{|z|=1} \frac{dz}{\sqrt{6z^2-5z+1}} = ?$$

where the square root is $\sqrt{2}$ at $z=1$.

sol (sketch)

2 zeros of $6z^2-5z+1$

① (see HW8. Problem 6) $\exists!$ function $f(z)$ (call $\sqrt{6z^2-5z+1}$) analytic in $\mathbb{C}-\{z\in\mathbb{R}\subset\mathbb{C} : \frac{1}{3}\le z\le\frac{1}{2}\}$ s.t.

$f(z)^2 = 6z^2-5z+1$, $f(1)=\sqrt{2}$

② Since $\sqrt{6z^2-5z+1} \sim \sqrt{6}\, z$ for large z, $\leftarrow$ $\varepsilon(z) = \sqrt{6z^2-5z+1} - \sqrt{6}\,z$ has the property $\frac{\varepsilon(z)}{z}\to 0$ as $z\to\infty$

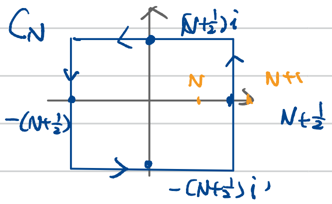
$$\int_{|z|=R} \frac{dz}{\sqrt{6z^2-5z+1}} \longrightarrow \frac{1}{\sqrt{6}}\int_{|z|=R}\frac{dz}{z} = \frac{2\pi i}{\sqrt{6}}$$

③ By Homotopy Thm,

$$\int_{|z|=1} \frac{dz}{\sqrt{6z^2-5z+1}} = \int_{|z|=R} \frac{dz}{\sqrt{6z^2-5z+1}} = \frac{2\pi i}{\sqrt{6}} \qquad \#$$

Sums (§11.2)

Example $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$



pf

Let C_N be the curve

The function

$$\frac{1}{z^2}\pi\cot\pi z = \frac{\pi}{z^2}\frac{\cos\pi z}{\sin\pi z}$$

has poles at $n \in \mathbb{Z}$

$$\left.\frac{\frac{\pi\cos\pi z}{z^2}}{\pi\cos\pi z}\right|_{z=n} = \frac{1}{n^2}$$

exer: $|\cot\pi z| < 2$ on C_N

$$\left|\int_{C_N}\frac{1}{z^2}\pi\cot\pi z\,dz\right| \le (8N+4)\,2\pi \max_{z\in C_N}\underbrace{\left|\frac{1}{z^2}\right|}_{\sim \frac{1}{N^2}} \longrightarrow 0 \text{ as } N\to\infty$$

$$\Rightarrow \int_{C_N}\frac{1}{z^2}\pi\cot\pi z\,dz = 2\pi i\left(\mathrm{Res}\left(\frac{\pi}{z^2}\cot\pi z\,;0\right) + \sum_{\substack{n=-N\\ n\neq 0}}^{N}\mathrm{Res}\left(\frac{\frac{\pi\cos\pi z}{z^2}}{\sin\pi z}\,;n\right)\right) \to 0$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{n^2} = -\frac{1}{2}\mathrm{Res}\left(\frac{\pi\cot\pi z}{z^2}\,;0\right)$$

Fact Laurent expansion of $\cot z$ is $\frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2z^5}{945} \cdots$

$$= -\frac{1}{2}\left(-\frac{\pi^2}{3}\right) = \frac{\pi^2}{6} \qquad \#$$

(Similarly, $\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$)

Remark (see wiki or Ch18)

- The Riemann zeta function $\zeta(z)$ is a meromorphic function with a simple pole at $z=1$ with residue 1 s.t. $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$ when RHS makes sense ($\mathrm{Re}(s)>1$)
- Fact: $\zeta(-2k) = 0 \quad \forall\, k\in\mathbb{N}$
- Conjecture (Riemann hypothesis): $\{\text{zeros of } \zeta(z)\} \setminus (-2\mathbb{N}) \subseteq \{z\in\mathbb{C} : \mathrm{Re}(z) = \frac{1}{2}\}$

Example $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

pf

Consider the same C_N, but a different function

$$\frac{1}{z^2}\pi\csc\pi z = \frac{\pi}{z^2\sin\pi z}$$

$$\left.\frac{\frac{\pi}{z^2}}{(\sin\pi z)'}\right|_{z=n} = \frac{(-1)^n}{n^2}$$

A similar argument shows that

$$\int_{C_N}\frac{\pi\csc\pi z}{z^2}\,dz = 2\pi i\left(\mathrm{Res}\left(\frac{\pi\csc\pi z}{z^2}\,;0\right) + \sum_{\substack{n=-N\\ n\neq 0}}^{N}\mathrm{Res}\left(\frac{\pi\csc\pi z}{z^2}\,;n\right)\right) \longrightarrow 0$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = -\frac{1}{2}\left(\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^n}{n^2}\right) = \frac{1}{2}\mathrm{Res}\left(\frac{\pi\csc\pi z}{z^2}\,;0\right) = \frac{\pi^2}{12} \qquad \#$$

Fact: $\frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \cdots$

Example $\sum_{k=0}^{n}\binom{n}{k}^2 = ?$

sol

Note that

$\binom{n}{k} =$ Coeff of z^k in $(1+z)^n$

$\binom{n}{k} =$ Coeff of z^{-k} in $(1+\frac{1}{z})^n$

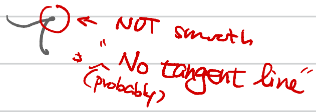
$\Rightarrow \sum_{k=0}^{n}\binom{n}{k}^2 =$ coeff of $z^k\cdot z^{-k} = 1$ in $(1+z)^n\cdot(1+\frac{1}{z})^n$

So

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \frac{1}{2\pi i}\int_{|z|=1}(1+z)^n\left(1+\frac{1}{z}\right)^n\cdot\frac{dz}{z} = \frac{1}{2\pi i}\int_{|z|=1}\frac{(1+z)^{2n}}{z^{n+1}}\,dz$$

$$= \text{Coeff of } z^n \text{ in } (1+z)^{2n} = \binom{2n}{n} \qquad \#$$

Ch 13-14 Conformal mapping



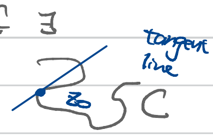
Conformal equivalence

Def 13.1

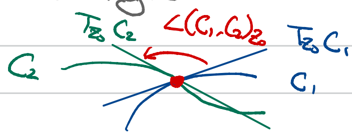
(regarding C as a subset in $\mathbb{C}$)

Let C be a curve in $\mathbb{C}$ and $z_0 \in C$. We say C is **smooth** at z_0 if $\exists$ parametrization $z : (-\varepsilon, \varepsilon) \to C$ s.t. $z(0) = z_0$, $z'(0) \neq 0$

In this case, the **tangent line** of C is the line $T_{z_0}C := \{ z_0 + t z'(0) : t \in \mathbb{R} \}$



Let C_1 and C_2 be 2 curves which are smooth and intersects at z_0. The **angle from C_1 to C_2 at z_0**, denoted $\angle(C_1, C_2)_{z_0}$, is defined as the angle measured counterclockwise from $T_{z_0}C_1$ to $T_{z_0}C_2$



Def 13.2

Suppose f is defined in a nbd of z_0.

f is said to be **conformal at z_0** if "f preserves angles at z_0".

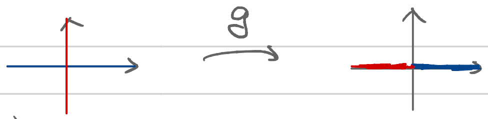
That is, for each pair of curves C_1, C_2 smooth at z_0, intersecting at z_0, one has $\angle(C_1, C_2)_{z_0} = f(\angle(C_1,C_2)_{z_0}) := \angle(f(C_1), f(C_2))_{f(z_0)}$

We say f is **conformal** in a region D if f is conformal at all points $z \in D$

Example

- $f(z) = z$ is conformal in $\mathbb{C}$
- $g(z) = z^2$ is NOT conformal at 0 :

  (But, in fact, g is conformal in $\mathbb{C} - \{0\}$)



Def 13.3

a. f is a **1-1** function in a region D if for every $z_1 \neq z_2$ in D, $f(z_1) \neq f(z_2)$

b. f is **locally 1-1** at z_0 if f is 1-1 in a nbd of z_0

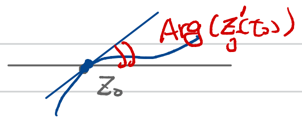
c. f is **locally 1-1 throughout a region D** if f is locally 1-1 at every $z \in D$

exer $g(z) = z^2$ is (i) NOT locally 1-1 at 0, (ii) NOT 1-1 in $\mathbb{C} - \{0\}$, but (iii) locally 1-1 in $\mathbb{C} - \{0\}$.

Thm 13.4 (c.f. Inverse Function Thm)

Suppose f is analytic at z_0 and $f'(z_0) \neq 0$. Then f is conformal and locally 1-1 at z_0

pf



① Let C_j be parametrized by $z_j(t) = x_j(t) + i y_j(t)$, $z_j(t_0) = z_0$, $j = 1, 2$

$\Rightarrow f(C_j)$ is parametrized by $\omega_j(t) = f(z_j(t))$

$\Rightarrow \mathrm{Arg}\, \omega_j'(t_0) = \mathrm{Arg}(f'(z_0) \cdot z_j'(t_0)) = \mathrm{Arg}(f'(z_0)) + \mathrm{Arg}(z_j'(t_0))$

$\Rightarrow \angle(C_1, C_2)_{z_0} = \mathrm{Arg}(z_2'(t_0)) - \mathrm{Arg}(z_1'(t_0))$

$= (\mathrm{Arg}\, f'(z_0) + \mathrm{Arg}\, z_2'(t_0)) - (\mathrm{Arg}\, f'(z_0) + \mathrm{Arg}\, z_1'(t_0))$

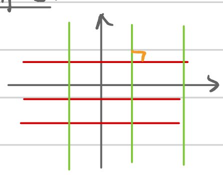
$= \angle(f(C_1), f(C_2))_{f(z_0)}$

$\Rightarrow f$ is conformal.

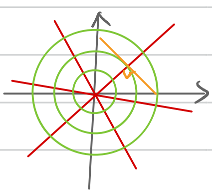
② f is locally 1-1 by the inverse function thm (recall the total differential $(Df(z_0))(z - z_0) = f'(z_0) \cdot (z - z_0)$) ※

see p170-171 for a different proof for locally 1-1.

Example 1



$\xrightarrow{e^z}$ Conformal, locally 1-1 (but not 1-1)



Example 2

Let $f(z) = z^2$. $\Rightarrow f'(z) = 2z \neq 0 \quad \forall z \neq 0 \quad \Rightarrow f$ is conformal, locally 1-1 (NOT 1-1) throughout $\mathbb{C} - \{0\}$ see p. 172 for picture

Def 13.5

Let $k \in \mathbb{N}$. We say f is a k-to-1 mapping of D_1 onto D_2 if $\forall \alpha \in D_2$, the eq $f(z) = \alpha$ has k roots (counting multiplicity) in D_1.

Example

Let $f(z) = z^k$, $k \in \mathbb{N}$, $\delta > 0$. Then $f: D(0;\delta) \longrightarrow D(0;\delta^k)$ is k-to-1.

Thm 13.7 (cf. Thm 13.4.)

Suppose f is analytic at z_0 with $f'(z_0) = 0$

If f is not constant in a nbd of z_0, then f is a k-to-1 mapping and f magnifies angles at z_0 by k in a nbd of z_0, where k is the least positive integer for which $f^{(k)}(z_0) \neq 0$

pf

① Without loss of generality, assume $f(z_0) = 0$.

By hypothesis, the Taylor expansion of f about z_0 is of the form

$$f(z) = (z-z_0)^k \underbrace{\left(a_k + a_{k+1}(z-z_0) + a_{k+2}(z-z_0)^2 + \cdots\right)}_{=: g(z)}$$

with $a_k = f^{(k)}(z_0)/k! \neq 0$

② Since $g(z_0) \neq 0$, $\exists \delta > 0$ s.t. $g(z) \neq 0 \;\; \forall z \in D(z_0;\delta)$ ← simply connected

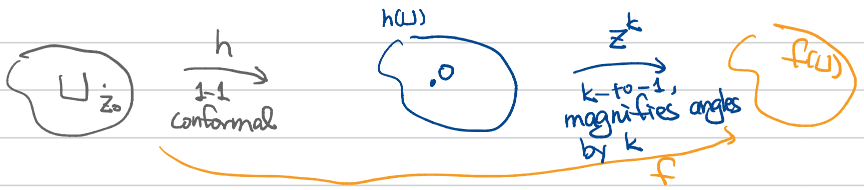
$\Rightarrow \exists \; g^{1/k}(z)$ analytic in $D(z_0;\delta)$ s.t. $(g^{1/k}(z))^k = g(z) \;\; \forall z \in D(z_0;\delta)$

Let

$$h(z) = (z-z_0)\, g^{1/k}(z) \quad \leftarrow \text{analytic}$$

$\Rightarrow f(z) = (h(z))^k$, $h(z_0) = 0$, $h'(z_0) = g^{1/k}(z_0) \neq 0$.

$\Rightarrow \exists$ nbd U of z_0 in $D(z_0;\delta)$ s.t. h is 1-1, conformal in U



#

Thm 13.8

Suppose f is a 1-1 analytic function in a region D. Then

a. f^{-1} exists and is analytic in $f(D)$

b. f and f^{-1} are conformal in D and $f(D)$, respectively.

pf

f is 1-1 $\overset{\text{Thm 13.7}}{\Longrightarrow}$ $f' \neq 0$ $\overset{\text{Prop 3.5}}{\Longrightarrow}$ f^{-1} is also analytic and $(f^{-1})' = \frac{1}{f'} \neq 0$

$\Rightarrow f$ and f^{-1} are both conformal. #

Def 13.9

a. A 1-1 analytic mapping is called a conformal mapping

b. Two regions D_1 and D_2 are conformally equivalent if $\exists$ conformal mapping $D_1 \overset{1-1}{\underset{\text{onto}}{\Rightarrow}} D_2$

Such a bijective conformal mapping is called a conformal equivalence or biholomorphism

Riemann Mapping Theorem (see §14.2)

Any two simply connected domains (except $\mathbb{C}$) are conformally equivalent