pf

Hence $\frac{f'(z)}{f(z)} = \frac{k}{z-a} + \frac{g'(z)}{g(z)}$ has a simple pole at a with residue k.

② Similarly, if f has a pole of order l at $z=b$, then

$$f(z) = (z-b)^{-l}\, h(z)$$

where $h(b) \neq 0$.

$\Rightarrow \frac{f'(z)}{f(z)} = -\frac{l}{z-b} + \frac{h'(z)}{h(z)}$ has a simple pole at b with residue $-l$.

③ By Residue Thm,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n} \operatorname{ord}(z_k)\cdot n(\gamma; z_k) - \sum_{j=1}^{m} \operatorname{ord}(p_j)\cdot n(\gamma; p_j) \qquad \#$$

Remark (p.136)

The above theorem is known as "Argument Principle" because

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \text{"}\frac{1}{2\pi i}(\log f(\gamma(b)) - \log f(\gamma(a)))\text{"} = \frac{1}{2\pi}\Delta \operatorname{Arg} f(\gamma)$$

"the argument $f(\gamma)$ covers" e.g. 

Def 10.4

γ is called a regular closed curve if γ is a simple closed piecewise C^1 curve with $n(\gamma, a) = 0$ or $n(\gamma, a) = 1$ $\forall a \notin \gamma$. In this case, we will call $\{a \in \mathbb{C} : n(\gamma, a) = 1\}$ the inside of γ and call $\{a \in \mathbb{C} : n(\gamma, a) = 0\}$ the outside of γ.

Rouché Thm (Thm 10.10)

Suppose that f and g are analytic inside and on a regular closed curve γ and that $|f(z)| > |g(z)|$ $\forall z \in \gamma$. Then

$\Rightarrow f \neq 0,\ f+g \neq 0$ on γ ($|f+g| \geq |f| - |g| > 0$) ⊛

$$Z_\gamma(f+g) = Z_\gamma(f) \quad \text{inside } \gamma,$$

where $Z_\gamma(f)$ is the number of zeros of f inside γ, counting multiplicities.

pf

① Note that if $f(z) = A(z)\,B(z)$, then $\frac{f'}{f} = \frac{A'}{A} + \frac{B'}{B}$

$$\Rightarrow \int_\gamma \frac{f'(z)}{f(z)}\,dz = \int_\gamma \frac{A'(z)}{A(z)}\,dz + \int_\gamma \frac{B'(z)}{B(z)}\,dz$$

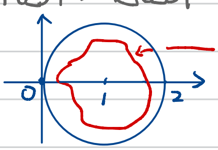
② Since $f+g = f\left(1+\frac{g}{f}\right)$, we have

Argument Principle; f analytic $\Rightarrow$ no pole

$$Z_\gamma(f+g) = \frac{1}{2\pi i}\int_\gamma \frac{(f+g)'}{\underbrace{f+g}_{\neq 0}}\,dz = \frac{1}{2\pi i}\left(\int_\gamma \frac{f'}{\underbrace{f}_{\neq 0}}\,dz + \int_\gamma \frac{(1+\frac{g}{f})'}{\underbrace{(1+\frac{g}{f})}_{\neq 0}}\,dz\right) \quad ⊛$$

$$= Z_\gamma(f) + \frac{1}{2\pi i}\int_\gamma \frac{(1+\frac{g}{f})'}{(1+\frac{g}{f})}\,dz$$

③ Since $|f(z)| > |g(z)|$ $\forall z \in \gamma$, we have $\left|\frac{g}{f}\right| < 1$



$(1+\frac{g}{f})(\gamma) \subseteq D(1;1)$

Since $D(1;1)$ is simply connected, by Thm 8.6, $\exists$ analytic branch h of $\log$ in $D(1;1)$ $\Rightarrow \int_\gamma \frac{(1+\frac{g}{f})'}{(1+\frac{g}{f})}\,dz = h\left(1+\frac{g}{f}\right)\Big|_{z=\gamma(a)}^{\gamma(b)} = 0$ (antiderivative)

So by ②, $Z_\gamma(f+g) = Z_\gamma(f)$ #

Example

Show that the polynomials

$$2z^{10} + 4z^2 + 1 \quad \text{and} \quad 2z^{10} - 4z^2 + 1$$

has exactly 2 zeros (counting multiplicities) in $|z| < 1$.

pf

Note that, on $|z| = 1$,

$$|4z^2| = 4 > 3 \geq |2z^{10} + 1|$$

By Rouché Thm,

$$Z_\gamma(2z^{10} + 4z^2 + 1) = Z_\gamma(2z^{10} - 4z^2 + 1) = Z_\gamma(4z^2) = 2 \qquad \#$$

Ch 11-12 Applications of Residue Thm

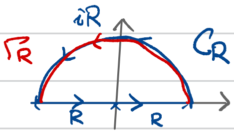
I. $\int_{-\infty}^{\infty} P(x)/Q(x)\,dx$

Example

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = ?$$

sol

① Let C_R be the closed curve



where R is large enough so that all the zeros of z^4+1 in $H^+ := \{z\in\mathbb{C} : \operatorname{Im} z \geq 0\}$

are inside C_R

Let Γ_R be $Re^{i\theta}$, $\theta\in[0,\pi]$. Then.

$$\int_{C_R} \frac{1}{z^4+1}\,dz = \int_{-R}^{R} \frac{1}{x^4+1}\,dx + \int_{\Gamma_R} \frac{1}{z^4+1}\,dz$$

Note that $\quad \swarrow \ |z^4+1| \geq |z^4|-1 = R^4-1 \Rightarrow \left|\frac{1}{z^4+1}\right| \leq \frac{1}{R^4-1}$

$$\left|\int_{\Gamma_R} \frac{1}{z^4+1}\,dz\right| \leq \pi\cdot R\cdot \frac{1}{R^4-1} \longrightarrow 0 \quad \text{as} \quad R\to\infty$$

So

$$\boxed{\int_{C_R} \frac{1}{z^4+1}\,dz \longrightarrow \int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx \quad \text{as} \quad R\to\infty}$$

② Let

$$Z^+(z^4+1) = \{z\in H^+ : z^4+1=0\}$$

Note that $Z^+(z^4+1)\cap\mathbb{R} = \phi$, and

$$Z^+(z^4+1) = \{e^{\frac{\pi}{4}i}, e^{\frac{3}{4}\pi i}\}$$

$$\operatorname{Res}\left(\frac{1}{z^4+1}; e^{\frac{\pi}{4}i}\right) = \frac{1}{4z^3}\Big|_{e^{\frac{\pi}{4}i}} = \frac{1}{4}e^{-\frac{3}{4}\pi i} = \frac{1}{4}\left(\frac{-1}{\sqrt{2}} + i\frac{-1}{\sqrt{2}}\right) = \frac{-1}{8}(\sqrt{2}+i\sqrt{2})$$

$$\operatorname{Res}\left(\frac{1}{z^4+1}; e^{\frac{3}{4}\pi i}\right) = \frac{1}{4z^3}\Big|_{e^{\frac{3}{4}\pi i}} = \frac{1}{4}e^{-\frac{9}{4}\pi i} = \frac{1}{4}\left(\frac{1}{\sqrt{2}} + i\frac{-1}{\sqrt{2}}\right) = \frac{1}{8}(\sqrt{2}-i\sqrt{2})$$

By Residue Thm (Thm 10.5),

$$\int_{C_R} \frac{1}{z^4+1}\,dz = 2\pi i\left(\operatorname{Res}\left(\frac{1}{z^4+1}; e^{\frac{\pi}{4}i}\right) + \operatorname{Res}\left(\frac{1}{z^4+1}; e^{\frac{3}{4}\pi i}\right)\right) = \frac{\pi i}{4}(-2i\sqrt{2}) = \frac{\sqrt{2}}{2}\pi$$

$$\longrightarrow \int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx \quad \text{as } R\to\infty \quad \text{by ①} \qquad \to \frac{\sqrt{2}}{2}\pi$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = \frac{\sqrt{2}}{2}\pi \quad \#$$

Remark

Step① applies to "$\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}dx$ with $\deg Q - \deg P \geq 2$ and $Q(x)\neq 0\ \forall x\in\mathbb{R}$"

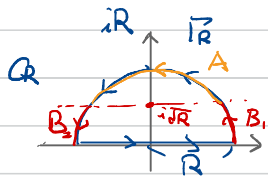
The same argument shows that

$$\boxed{\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{z\in Z^+(Q)} \operatorname{Res}\left(\frac{P}{Q}; z\right)}$$

II. $\int_{-\infty}^{\infty} R(x)\cos x\,dx$ and $\int_{-\infty}^{\infty} R(x)\sin x\,dx$

Example $\quad \displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx = ? \qquad \int_{-\infty}^{\infty} \frac{\sin x}{x^2+1}\,dx = ?$

sol



Let Γ_R : $Re^{i\theta}$, $\theta\in[0,\pi]$

A : $Re^{i\theta}$, $\theta\in[\varphi, \pi-\varphi]$, $\varphi = \sin^{-1}\frac{\sqrt{R}}{R}$

B_1 : $Re^{i\theta}$, $\theta\in[0,\varphi]$; $\quad B_2$: $Re^{i\theta}$, $[\pi-\varphi, \pi]$

Consider

$$\int_{C_R} \frac{1}{z^2+1}e^{iz}\,dz = \int_{-R}^{R} \frac{1}{x^2+1}e^{ix}\,dx + \underbrace{\int_{\Gamma_R} \frac{1}{z^2+1}e^{iz}\,dz}_{=\int_{B_1}+\int_A+\int_{B_2}}$$

$|e^{iz}| = e^{-\text{Im}z} \leq e^{-\sqrt{R}}$

$\sqrt{R} = R\sin\varphi$

$\frac{\sqrt{R}}{R\varphi} = \frac{\sin\varphi}{\varphi} \geq \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$

$\Rightarrow R\varphi \leq \frac{\pi}{2}\sqrt{R}$

<u>sol</u>

① (a) $\left|\int_A \frac{1}{z^2+1} e^{iz}dz\right| \leq \frac{1}{R^2-1} e^{-\sqrt{R}} R(\pi-2\varphi) \leq \frac{R}{R^2-1} e^{-\sqrt{R}}\pi \longrightarrow 0$ as $R\to\infty$

(b) $\left|\int_{B_1} \frac{1}{z^2+1} e^{iz}dz\right| \leq \frac{1}{R^2-1}\, 1\, R\cdot\varphi \leq \frac{1}{R^2-1}\frac{\pi}{2}\sqrt{R} \longrightarrow 0$ as $R\to\infty$

$\left|\int_{B_2} \frac{1}{z^2+1} e^{iz}dz\right| \leq \frac{1}{R^2-1}\frac{\pi}{2}\sqrt{R} \longrightarrow 0$ as $R\to\infty$

So $\int_{\Gamma_R} \frac{1}{z^2+1} e^{iz}dz \longrightarrow 0$ as $R\to\infty$

$\Rightarrow \int_{C_R} \frac{1}{z^2+1} e^{iz}dz \longrightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}dx = \int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{x^2+1}dx$

② By <u>Residue Thm</u>, for R large enough

$$\int_{C_R} \frac{e^{iz}}{z^2+1}dz = 2\pi i\,\text{Res}\left(\frac{e^{iz}}{z^2+1}; i\right) = 2\pi i\cdot\left.\frac{e^{iz}}{2z}\right|_{z=i} = 2\pi i\,\frac{e^{-1}}{2i} = \pi e^{-1}$$

$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}dx = \text{Re}(\pi e^{-1}) = \pi e^{-1}$, $\quad \int_{-\infty}^{\infty} \frac{\sin x}{x^2+1}dx = \text{Im}(\pi e^{-1}) = 0$ #

<u>Remark</u>

- The above method works for $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\cos x\,dx$, $\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\sin x\,dx$ when $\deg Q > \deg P$, $Q(x)\neq 0\ \forall x\in\mathbb{R}$
- If $Q(x)$ and $\cos x$ (resp. $Q(x)$ and $\sin x$) have a common simple zero, one can modify the method by considering $\frac{P(x)}{Q(x)}(e^{iz}\pm i)$ (resp. $\frac{P(x)}{Q(x)}(e^{iz}\pm 1)$)

<u>Example</u> $\int_{-\infty}^{\infty} \frac{\sin x}{x}dx = ?$

<u>sol</u>

also $=0$ at $x=0$

Note that $\frac{\sin x}{x} = \text{Im}\,\frac{e^{ix}}{x} = \text{Im}\,\frac{e^{ix}-1}{x}$ and

$\frac{e^{iz}-1}{z} = \frac{1}{z}\sum_{n=1}^{\infty}\frac{1}{n!}(iz)^n = \sum_{n=1}^{\infty}\frac{(i)^n}{n!}z^{n-1} = \sum_{n=0}^{\infty}\frac{i^{n+1}}{(n+1)!}z^n = i - \frac{z}{2} + \cdots$ "is" entire (0 is removable singularity)

$\Rightarrow \int_{C_R}\frac{e^{iz}-1}{z}dz = 0 = \int_{-R}^{R}\frac{e^{ix}-1}{x}dx + \int_{\Gamma_R}\frac{e^{iz}-1}{z}dz$

$\Rightarrow \int_{-R}^{R}\frac{e^{ix}-1}{x}dx = \int_{\Gamma_R}\frac{1-e^{iz}}{z}dz = \int_{\Gamma_R}\frac{1}{z}dz - \int_{\Gamma_R}\frac{e^{iz}}{z}dz$

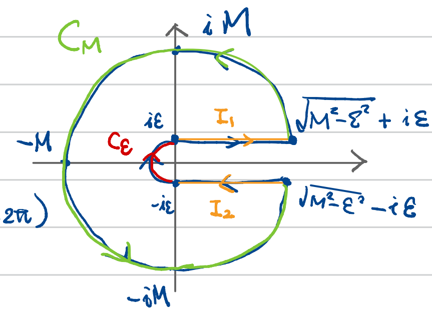
$\int_{\Gamma_R}\frac{1}{z}dz = \int_0^{\pi}\frac{1}{Re^{i\theta}}Rie^{i\theta}d\theta = \pi i$

Note that $\int_{\Gamma_R}\frac{e^{iz}}{z}dz \to 0$ as $R\to\infty$ ← similar as the previous example

So $\int_{-\infty}^{\infty}\frac{e^{ix}-1}{x}dx = \pi i \Rightarrow \int_{-\infty}^{\infty}\frac{\sin x}{x}dx = \text{Im}\left(\int_{-\infty}^{\infty}\frac{e^{ix}-1}{x}dx\right) = \pi.$ #

III. $\int_0^{\infty} P(x)/Q(x)\,dx$

<u>Example</u> $\int_0^{\infty}\frac{dx}{1+x^3} = ?$



<u>sol</u>

$C_\varepsilon \cup I_1 \cup I_2 \cup C_M$

Let $K_{\varepsilon,M}$ be the curve defined as in the diagram

Note $\log z$ can be defined on $\mathbb{C}-\mathbb{R}_{\geq 0} \supseteq K_{\varepsilon M}$ (Choose $0<\text{Arg}z<2\pi$)

Consider $\int_{K_{\varepsilon M}}\frac{1}{1+z^3}\log z\,dz$

① (a) Since $\frac{1}{1+z^3}$ is continuous at $z=0$, $\exists\ \delta_0>0$, $A>0$ s.t.

$\left|\frac{1}{1+z^3}\right| \leq A \quad \forall\ |z|<\delta_0$

Also note that for $z\in C_\varepsilon$, $|\log z| = |\log \varepsilon e^{i\theta}| = |\log\varepsilon + i\theta| \leq |\log\varepsilon| + 2\pi$

$\Rightarrow \forall\ \varepsilon<\delta_0$,

$\left|\int_{C_\varepsilon}\frac{1}{1+z^3}\log z\,dz\right| \leq (\pi\cdot\varepsilon)\cdot A(|\log\varepsilon|+2\pi) \longrightarrow 0$ as $\varepsilon\to 0.$

Sol

① (ii) For $z \in C_M$,

$$\left| \frac{1}{z^3+1} \log z \right| \le \frac{1}{M^3-1} \cdot (\log M + 2\pi)$$

$$\Rightarrow \left| \int_{C_M} \frac{1}{z^3+1} \log z \, dz \right| \le 2\pi M \cdot \frac{1}{M^3-1} (\log M + 2\pi) \longrightarrow 0 \quad \text{as } M \to \infty$$

(iii) $$\int_{I_1} \frac{1}{z^3+1} \log z \, dz = \int_0^{\sqrt{M^2-\varepsilon^2}} \frac{1}{(i\varepsilon + t)^3 + 1} \log(i\varepsilon + t) \, dt$$

$i\varepsilon + t$

$0 \le t \le \sqrt{M^2-\varepsilon^2}$

$0 < \delta < \sqrt{M^2-\varepsilon^2}$

$$= \int_0^\delta \frac{1}{(i\varepsilon+t)^3+1} \log(i\varepsilon+t) \, dt + \int_\delta^M \frac{1}{(i\varepsilon+t)^3+1} \log(i\varepsilon+t) \, dt \longrightarrow \int_\delta^M \frac{1}{t^3+1} \log t \, dt \quad \text{as } \varepsilon \to 0$$

as $\delta \to 0$: $\int_0^M \frac{1}{t^3+1} \log t \, dt$

because $\frac{1}{z^3+1} \log z$ is uniformly continuous on

δ M

$$- \int_{\sqrt{M^2-\varepsilon^2}}^M \frac{1}{(i\varepsilon+t)^3+1} \log(i\varepsilon+t) \, dt$$

if $M > 2$, $\varepsilon < 1$

when $\sqrt{\varepsilon^2+\delta^2} < 1/2$

$$\left| \int_0^\delta - \right| \le \delta \cdot \frac{1}{1-(\varepsilon^2+\delta^2)^{3/2}} \left(|\log \sqrt{\varepsilon^2+\delta^2}| + \underbrace{|\mathrm{Arg}\, i\varepsilon + t|}_{\le 2\pi} \right) \le \delta \frac{1}{1-(\frac{1}{2})^{3/2}} (|\log \delta| + 2\pi) \longrightarrow 0 \quad \text{as } \delta, \varepsilon \to 0$$

$$\left| \int_{\sqrt{M^2-\varepsilon^2}}^M - \right| \le (M - \sqrt{M^2-\varepsilon^2}) \cdot \frac{1}{(M^2+1)^{3/2}+1} \left(\log \sqrt{M^2+1} + 2\pi \right) \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$

← Maybe leave this part as exercises

So $$\int_{I_1} \frac{1}{z^3+1} \log z \, dz \longrightarrow \int_0^M \frac{1}{t^3+1} \log t \, dt \quad \text{as } \varepsilon \to 0$$

$$\longrightarrow \int_0^\infty \frac{1}{t^3+1} \log t \, dt \quad \text{as } M \to \infty$$

Similarly,

$$\int_{I_2} \frac{1}{z^3+1} \log z \, dz \longrightarrow - \int_0^M \frac{1}{t^3+1} (\log t + 2\pi i) \, dt \quad \text{as } \varepsilon \to 0$$

$$\longrightarrow - \int_0^\infty \frac{1}{t^3+1} \log t \, dt - 2\pi i \int_0^\infty \frac{1}{t^3+1} \, dt \quad \text{as } M \to \infty$$

Note : θ $re^{i\theta} = t + i\varepsilon$ $\log(t+i\varepsilon) = \log r + i\theta$ $\to$ "$\log r$" $+ i2\pi$ when $\varepsilon \to 0$

Therefore,

$$\int_{K_{M,\varepsilon}} \frac{1}{z^3+1} \log z \, dz \longrightarrow -2\pi i \int_0^\infty \frac{1}{t^3+1} \, dt \quad \text{as } M \to \infty, \ \varepsilon \to 0$$

② Note that z^3+1 has 3 zeros : $e^{i\pi/3}$, $e^{i\pi}$, $e^{i5\pi/3}$, and

$$\mathrm{Res}\left(\frac{\log z}{1+z^3} ; e^{i\frac{\pi}{3}}\right) = \frac{\log z}{3z^2}\Big|_{z=e^{i\frac{\pi}{3}}} = i\frac{\pi}{3} \cdot \frac{1}{3} e^{i\frac{-2}{3}\pi} = \frac{i\pi}{9}\left(\frac{-1}{2} - \frac{\sqrt{3}}{2} i\right)$$

$$\mathrm{Res}\left(\frac{\log z}{1+z^3} ; e^{i\pi}\right) = \frac{\log z}{3z^2}\Big|_{z=e^{i\pi}} = i\pi \ \frac{1}{3} e^{i(-2\pi)} = \frac{i\pi}{3}$$

$$\mathrm{Res}\left(\frac{\log z}{1+z^3} ; e^{i\frac{5\pi}{3}}\right) = \frac{\log z}{3z^2}\Big|_{z=e^{i\frac{5\pi}{3}}} = i\frac{5\pi}{3} \ \frac{1}{3} e^{i\left(\frac{-10\pi}{3}\right)} = \frac{5\pi}{9} i \left(\frac{-1}{2} + \frac{\sqrt{3}}{2} i\right)$$

So, by Residue Thm, for M large enough, ε small enough,

$$\int_{K_{M,\varepsilon}} \frac{1}{z^3+1} \log z \, dz = 2\pi i \left(\frac{i\pi}{9}\left(\cancel{\frac{-1}{2}} - \frac{\sqrt{3}}{2} i\right) + \cancel{\frac{i\pi}{3}} + \frac{5\pi i}{9}\left(\cancel{\frac{-1}{2}} + \frac{\sqrt{3}}{2} i\right) \right) = -\frac{2\pi}{9}\sqrt{3} \cdot (2\pi i)$$

$$\xrightarrow{①} -2\pi i \int_0^\infty \frac{1}{t^3+1} \, dt$$

$$\Rightarrow \int_0^\infty \frac{1}{t^3+1} \, dt = \frac{2\pi}{9}\sqrt{3} \qquad \#$$

Remark

- The above method works for $\int_a^\infty \frac{P(x)}{Q(x)} \, dx$ ← $\frac{P(z)}{Q(z)} \log(z-a)$, $\deg Q - \deg P \ge 2$, $Q(x) \ne 0 \ \forall x \in [a, \infty)$
- This method also applies to $\int_0^\infty \frac{x^{\alpha-1}}{P(x)} \, dx$, $\deg P \ge 1$, $0 < \alpha < 1$, $P(x) \ne 0 \ \forall x \in [0, \infty)$

Example

$$\int_0^\infty \frac{dx}{\sqrt{x}(1+x)} = ?$$

Sol

(Choose $0 < \mathrm{Arg}\, z < 2\pi$)

Let $K_{M,\varepsilon}$ be as in the previous example. Consider $\sqrt{z} = e^{\frac{1}{2}\log z}$ is analytic in $\mathbb{C} - \mathbb{R}_{\ge 0}$

Similar as in the previous example, one can show

$$\int_{C_\varepsilon} \frac{dz}{\sqrt{z}(1+z)} \to 0 \quad \text{as } \varepsilon \to 0, \qquad \int_{C_M} \frac{dz}{\sqrt{z}(1+z)} \to 0 \quad \text{as } M \to \infty$$

$$\int_{I_1 \cup I_2} \frac{1}{\sqrt{z}(1+z)} \, dz \longrightarrow \left(1 - e^{\frac{-1}{2}(2\pi i)}\right) \int_0^\infty \frac{1}{\sqrt{x}(1+x)} \, dx = (1 - \underbrace{e^{-\pi i}}_{-1}) \int_0^\infty \frac{1}{\sqrt{x}(1+x)} \, dx \quad \text{as } \varepsilon \to 0, \ M \to \infty$$

By Residue Thm, for M large enough, ε small enough,

$$\int_{K_{\varepsilon M}} \frac{1}{\sqrt{z}(1+z)} \, dz = 2\pi i \, \mathrm{Res}\left(\frac{1}{\sqrt{z}(1+z)} ; -1\right) = 2\pi i \cdot \left(\frac{z^{-\frac{1}{2}}}{1} \Big|_{z=-1} \right) = 2\pi i \, \underbrace{e^{-\frac{\pi}{2} i}}_{= -i} = 2\pi$$

$$\longrightarrow 2 \int_0^\infty \frac{1}{\sqrt{x}(1+x)} \, dx$$

So $$\int_0^\infty \frac{1}{\sqrt{x}(1+x)} \, dx = \pi \qquad \#$$