Ch 10 Residues

Evaluation of residues

Def 10.1

If $f(z)=\sum_{k=-\infty}^{\infty} C_k (z-z_0)^k$ in a deleted nbd of z_0, C_{-1} is called the residue of f at z_0.
We use the notation $C_{-1} = \mathrm{Res}(f; z_0)$.

Remark (p. 129–130)

(i) If f has a simple pole at z_0, i.e.,
$$f(z)=\frac{A(z)}{B(z)}$$
where A and B are analytic at z_0, $A(z_0)\neq 0$ and B has a simple zero at z_0, then
$$C_{-1} = \lim_{z\to z_0}(z-z_0)f(z) = \frac{A(z_0)}{B'(z_0)}$$
(ii) If f has a pole of order k at z_0,
$$C_{-1} = \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left((z-z_0)^k f(z)\right)\Big|_{z=z_0}$$
pf of (i), (ii): Compute by Laurent expansions (use the remarks on p124–125)
(iii) In general, to obtain $\mathrm{Res}(f;z_0)$, one just compute the Laurent expansion of f at z_0.

Example

$\sin 0 = 0,\ (\sin z)'|_{z=0} = 1 \neq 0 \Rightarrow$ simple pole

(i) $\mathrm{Res}(\csc z; 0) = \mathrm{Res}(\frac{1}{\sin z}; 0) = \frac{1}{\cos 0} = 1$

(ii) $\mathrm{Res}(\frac{1}{z^4-1}; i) = \frac{1}{4i^3} = \frac{i}{4}$ (simple pole)

(iii) $\mathrm{Res}(\frac{1}{z^3}; 0) = 0$

(iv) $\mathrm{Res}(\sin\frac{1}{z-1}; 1) = 1$ because $\sin\frac{1}{z-1} = \frac{1}{z-1} - \frac{1}{3!(z-1)^3} + \frac{1}{5!(z-1)^5} - \cdots$

(v) $\mathrm{Res}(\sin\frac{1}{z-1}; i) = 0$ because $\sin\frac{1}{z-1}$ is analytic at i

Winding number

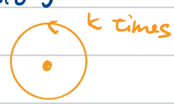
Def 10.2

Suppose that γ is a closed and that $a \notin \gamma$. Then
$$n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$$
is called the winding number of γ around a.

Example

① If $\gamma(\theta) = z_0 + re^{i\theta}$, $0\le\theta\le 2\pi$, then,
$$n(\gamma,a) = \begin{cases} 1 & \text{if } a\in D(z_0;r) \\ 0 & \text{if } a\notin \overline{D(z_0;r)} \end{cases}$$
(Cauchy Integral Formula, Thm 5.3)
(Closed Curve Thm, Thm 8.6)

② If $\gamma(\theta) = a + re^{ik\theta}$, $0\le\theta\le 2\pi$, $k\in\mathbb{Z}\setminus\{0\}$, then
$$n(\gamma,a) = \frac{1}{2\pi i}\int_0^{2\pi}\frac{1}{re^{ik\theta}}\, r\cdot ik e^{ik\theta}\, d\theta = k$$



Thm 10.3

For any closed curve γ and $a\notin\gamma$, $n(\gamma,a)$ is an integer.

pf

Suppose γ is given by $z(t)$, $0\le t\le 1$, and set
$$F(s) = \int_0^s \frac{z'(t)}{z(t)-a}\,dt \qquad 0\le s\le 1$$
$F(1) = 2\pi i\cdot n(\gamma,a)$

$\Rightarrow F'(s) = z'(s)/(z(s)-a)$ (Apply Fundamental Thm of Calculus to Re and Im parts respectively)

$\Rightarrow \frac{d}{ds}[(z(s)-a)e^{-F(s)}] = z'(s)\cdot e^{-F(s)} + (z(s)-a)\cdot(-F'(s))e^{-F(s)} = 0$

$\Rightarrow (z(s)-a)e^{-F(s)} = \text{constant} = (z(0)-a)e^{-F(0)} = z(0)-a$

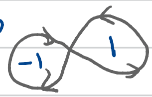
$\Rightarrow e^{F(s)} = (z(s)-a)/(z(0)-a)$

Since γ is closed, $e^{F(1)} = \frac{z(1)-a}{z(0)-a} = 1$ ($z(1) = z(0)$)

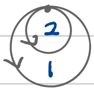
$\Rightarrow F(1) = 2\pi k i$ for some $k\in\mathbb{Z}$ $\Rightarrow n(\gamma,a) = \frac{1}{2\pi i}F(1) = k$ #

Example

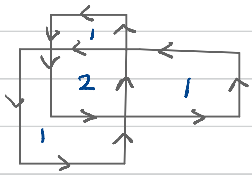
① $n=0$



② $n=0$



③ $n=0$



Residue Thm and applications

Thm 10.5 (Cauchy's Residue Thm)

Suppose f is analytic in a simply connected domain D except for isolated singularities at $z_1, z_2, \cdots, z_m$.

Let γ be a closed piecewise C^1 curve not intersecting any of the singlarities.

Then
$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{m} n(\gamma, z_k)\,\mathrm{Res}(f; z_k)$$

pf

Let $P_k\left(\frac{1}{z-z_k}\right)$ be the principal part of the Laurent expansion at z_k.

$$\Rightarrow \quad g(z) := \begin{cases} f(z) - P_1\left(\frac{1}{z-z_1}\right) - \cdots - P_k\left(\frac{1}{z-z_m}\right) & \text{if } z \neq z_1, \cdots, z_m \\ \lim_{z\to z_k} g(z) & \text{if } z = z_k \end{cases}$$

is an analytic function in D. (By Remark (p.124–125), $z_1, \cdots, z_m$ are removable singularitise of $\cdot$)

By Closed Curve Thm (Thm 8.6)
$$\int_\gamma g(z)\,dz = 0$$
$$\Rightarrow \int_\gamma f(z)\,dz = \sum_{k=1}^{m} \int_\gamma P_k\left(\frac{1}{z-z_k}\right) dz$$

Furthermore, if
$$P_k\left(\frac{1}{z-z_k}\right) = \frac{C_{-1}}{z-z_k} + \frac{C_{-2}}{(z-z_k)^2} + \cdots$$

$$\int_\gamma \frac{1}{(z-z_m)^j}\,dz = \left.\frac{(z-z_m)^{1-j}}{1-j}\right|_{z=\gamma(0)}^{\gamma(1)} = 0 \quad \text{for } j \neq 1$$

then
$$\int_\gamma P_k\left(\frac{1}{z-z_k}\right) dz = C_{-1}\int_\gamma \frac{1}{z-z_k}\,dz = \mathrm{Res}(f; z_k)\cdot 2\pi i \cdot n(\gamma, z_k)$$
$$\Rightarrow \int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{m} n(\gamma, z_k)\cdot \mathrm{Res}(f; z_k)$$
#

Example
$$\int_{|z|=1} \sin\frac{1}{z}\,dz \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{z}\right)^{2n+1}$$
$$= 2\pi i\,\mathrm{Res}\left(\sin\frac{1}{z}; 0\right)$$
$$= 2\pi i$$

Def 10.7 (holomorphic $\doteqdot$ analytic)

We say f is meromorphic in a domain D if f is analytic in D except at isolated poles.

Argument Principle (Thm 10.8, Cor 10.9)

Let f be a meromorphic function in a simply connected domain D with poles $p_1, \cdots, p_m$, and zeros $z_1, \cdots, z_n$. Suppose

$\mathrm{ord}(p_j) =$ the order of the pole p_j

$\mathrm{ord}(z_k) =$ the order of the zero z_k

If γ is a closed PC' curve in D, not passing through $p_1, \cdots, p_m$, $z_1, \cdots, z_m$, then
$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n} \mathrm{ord}(z_k)\cdot n(\gamma; z_k) - \sum_{j=1}^{m} \mathrm{ord}(p_j)\cdot n(\gamma; p_j)$$

pf

① If f has a zero of order k at $z=a$, then
$$f(z) = (z-a)^k g(z)$$
where $g(a) \neq 0$, so
$$f'(z) = k(z-a)^{k-1} g(z) + (z-a)^k g'(z)$$

pf

Hence $\dfrac{f'(z)}{f(z)} = \dfrac{k}{z-a} + \dfrac{g'(z)}{g(z)}$ has a simple pole at a with residue k.

② Similarly, if f has a pole of order ℓ at $z=b$, then

$$f(z) = (z-b)^{-\ell}\, h(z)$$

where $h(b) \neq 0$.

$\Rightarrow \dfrac{f'(z)}{f(z)} = -\dfrac{\ell}{z-b} + \dfrac{h'(z)}{h(z)}$ has a simple pole at b with residue $-\ell$.

③ By Residue Thm,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n} \operatorname{ord}(z_k)\cdot n(\gamma; z_k) - \sum_{j=1}^{m} \operatorname{ord}(p_j)\cdot n(\gamma; p_j) \qquad \#$$

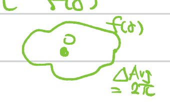
Remark (p.136)

The above theorem is known as "Argument Principle" because

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \text{"}\frac{1}{2\pi i}\big(\log f(\gamma(b)) - \log f(\gamma(a))\big)\text{"} = \frac{1}{2\pi}\,\Delta \operatorname{Arg} f(\gamma)$$

"the argument $f(\gamma)$ covers"

e.g.



Def 10.4

γ is called a regular closed curve if γ is a simple closed piecewise C' curve with $n(\gamma, a) = 0$ or $n(\gamma, a) = 1$ $\forall a \notin \gamma$. In this case, we will call $\{a \in \mathbb{C} : n(\gamma, a) = 1\}$ the inside of γ and call $\{a \in \mathbb{C} : n(\gamma, a) = 0\}$ the outside of γ.

Rouché Thm (Thm 10.10)

Suppose that f and g are analytic inside and on a regular closed curve γ and that $|f(z)| > |g(z)| \geq 0$ $\forall z \in \gamma$. Then $\Rightarrow g \neq 0,\ f \cdot g \neq 0$ on γ

$$Z(f+g) = Z(f) \quad \text{inside } \gamma,$$

where $Z(f)$ is the number of zeros of f inside γ, counting multiplicities.

pf

① Note that if $f(z) = A(z)\,B(z)$, then $\dfrac{f'}{f} = \dfrac{A'}{A} + \dfrac{B'}{B}$

$$\Rightarrow \int_\gamma \frac{f'(z)}{f(z)}\,dz = \int_\gamma \frac{A'(z)}{A(z)}\,dz + \int_\gamma \frac{B'(z)}{B(z)}\,dz$$

② Since $f+g = f\left(1+\frac{g}{f}\right)$, we have

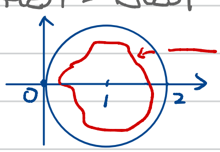
Argument Principle

$$Z(f+g) = \frac{1}{2\pi i}\int_\gamma \frac{(f+g)'}{f+g}\,dz = \frac{1}{2\pi i}\left(\int_\gamma \frac{f'}{f}\,dz + \int_\gamma \frac{\left(1+\frac{g}{f}\right)'}{\left(1+\frac{g}{f}\right)}\,dz\right)$$

(f analytic $\Rightarrow$ no pole)

$$= Z(f) + \frac{1}{2\pi i}\int_\gamma \frac{\left(1+\frac{g}{f}\right)'}{\left(1+\frac{g}{f}\right)}\,dz$$

③ Since $|f(z)| > |g(z)|$ $\forall z \in \gamma$, we have $\left|\frac{g}{f}\right| < 1$



$\left(1+\frac{g}{f}\right)(\gamma) \subseteq D(1;1)$

Since $D(1;1)$ is simply connected, by Thm 8.6, $\exists$ analytic branch h of $\log$ in $D(1;1)$ $\Rightarrow \displaystyle\int_\gamma \frac{\left(1+\frac{g}{f}\right)'}{\left(1+\frac{g}{f}\right)}\,dz = h\left(1+\frac{g}{f}\right)\Big|_{z=\gamma(a)}^{\gamma(b)} = 0$

anti derivative

So by ②, $Z(f+g) = Z(f)$ $\quad \#$

Example

Show that the polynomials

$$2z^{10} + 4z^2 + 1 \quad \text{and} \quad 2z^{10} - 4z^2 + 1$$

has exactly 2 zeros (counting multiplicities) in $|z| < 1$.

pf

Note that, on $|z| = 1$,

$$|4z^2| = 4 > 3 \geq |2z^{10} + 1|$$

By Rouché Thm,

$$Z(2z^{10} + 4z^2 + 1) = Z(2z^{10} - 4z^2 + 1) = Z(4z^2) = 2 \qquad \#$$