Ch9 Isolated singularities of an analytic function

3 types of isolated singularities

Def 9.1 & 9.2

A deleted neighborhood of $z_0 \in \mathbb{C}$ is a set of the form $D'(z_0;\varepsilon) = \{z \in \mathbb{C} : 0 < |z-z_0| < \varepsilon\}$ $= D(z_0;\varepsilon) \sim \{z_0\}$

A function f is said to have an isolated singularity at z_0 if f is analytic in a deleted nbd of z_0 but is NOT analytic at z_0

Suppose f has an isolated singularity at z_0

(i) If $\exists$ function g s.t. (a) g is analytic at z_0 (b) $f \equiv g$ in some deleted nbd of z_0, then we say f has a removable singularity at z_0

(ii) If $\exists$ A, B s.t. (a) A, B are analytic at z_0 (b) $A(z_0) \neq 0$, $B(z_0) = 0$ (c) $f \equiv \frac{A}{B}$ in some deleted nbd of z_0, then we say f has a pole at z_0

If B has a zero of order k at z_0 (i.e. $B(z_0) = B'(z_0) = \cdots = B^{(k-1)}(z_0) = 0$, $B^{(k)}(z_0) \neq 0$), then we say f has a pole of order k at z_0.

(iii) If f has neither a removable singularity nor a pole at z_0, then we say f has an essential singularity at z_0

Example (p.117)

(i) $f(z) := \begin{cases} \sin z, & z \neq 2 \\ 0, & z = 2 \end{cases}$ has a removable singularity at $z = 2$

(ii) $g(z) = \frac{1}{z-3}$ has a pole of order 1 at $z = 3$

(iii) $\exp(1/z)$ has an essential singularity at $z = 0$

Remark

By Thm 7.7, if f is analytic in $D'(z_0;\varepsilon)$ $\varepsilon>0$ and continuous at z_0, then f is analytic in $D(z_0;\varepsilon)$

So if f has an isolated singularity at z_0, then f must be discontinuous at z_0.

Removable singularity

Thm 9.3 (Riemann's Principle of Removable Singularities)

If f has an isolated singularity at z_0 and if $\lim_{z\to z_0} (z-z_0) f(z) = 0$, then the singularity is removable.

pf

Consider

$$h(z) = \begin{cases} (z-z_0) f(z) & z \neq z_0 \\ 0 & z = z_0 \end{cases}$$

By hypothesis, h is continuous at z_0. Since h, like f, is analytic in a deleted nbd of z_0, it follows that h is analytic at z_0 (continuous, analytic except one point, Thm 7.7)

Since $h(z_0) = 0$,

$$g(z) = \begin{cases} \frac{h(z)}{z-z_0} & \text{for } z \neq z_0 \\ h'(z_0) & z = z_0 \end{cases}$$

is analytic at z_0 and $g(z) = f(z)$ $\forall z \neq z_0$ #

Cor 9.4

If f is bounded in a deleted nbd of an isolated singularity, the singularity is removable.

(pf: Assume z_0 is the removable singularity. bdd $\Rightarrow$ $f(z)(z-z_0) \to 0$)

Pole

Thm 9.5

If f is analytic in a deleted nbd of z_0 and if $\exists\, k \in \mathbb{N}$ s.t.

$$\lim_{z\to z_0} (z-z_0)^k f(z) \neq 0 \quad \text{but} \quad \lim_{z\to z_0} (z-z_0)^{k+1} f(z) = 0,$$

then f has a pole of order k at z_0.

pf

If we set

$$g(z) = \begin{cases} (z-z_0)^{k+1} f(z), & z \neq z_0 \\ 0, & z = z_0 \end{cases}$$

then g is continuous and hence analytic at z_0 (Thm 7.7).

Furthermore, since $g(z_0)=0$,

$$A(z) = \begin{cases} \frac{g(z)}{z-z_0} = (z-z_0)^k f(z), & z \neq z_0 \\ g'(z_0), & z = z_0 \end{cases}$$

is analytic at z_0.

By assumption, $A(z_0) = \lim_{z\to z_0} (z-z_0)^k f(z) \neq 0$

$\Rightarrow \quad f(z) = \frac{A(z)}{(z-z_0)^k} \quad \forall z \neq z_0$

So f has a pole of order k. #

Remark

By Thm 9.3 and Thm 9.5, "$\nexists$ pole of order $\frac{q}{p} \in \mathbb{Q} \setminus \mathbb{Z}$" For example,

① if $|f(z)| \le \frac{1}{\sqrt{|z|}}$ in a deleted nbd of 0, then $zf(z) \to 0$ as $z \to 0$

$\Rightarrow$ 0 is a removable singularity by Thm 9.3 ($\nexists$ pole of order $\frac{1}{2}$)

② if $|f(z)| \le \frac{1}{|z|^{5/2}}$ in a deleted nbd of 0, then $z^2 f(z)$ has a removable singularity at 0 $\Rightarrow$ f has a pole of order at most 2 at 0 ($\nexists$ pole of order $\frac{5}{2}$)

Essential singularity

By Thm 9.3, Thm 9.5, if f has an essential singularity at z_0, then $\forall N \in \mathbb{N}$

$$(z-z_0)^N f(z) \not\to 0 \quad \text{as } z \to z_0$$

However, $f(z) \not\to \infty$ as $z \to z_0$. In fact, we have

Casorati–Weierstrass Thm (Thm 9.6)

If f is analytic in a deleted nbd D of z_0 and has an essential singularity at z_0, then the range $R = f(D)$ is dense in $\mathbb{C}$

pf

Assume R is not dense. Then $\exists\, \omega \in \mathbb{C}$, $\delta > 0$ s.t. $D(\omega; \delta) \cap R = \phi$, i.e. $|f(z)-\omega| \ge \delta \ \forall z \in D$

$\Rightarrow \quad \left|\frac{1}{f(z)-\omega}\right| \le \frac{1}{\delta} \quad \forall z \in D$

By Cor 9.4, $\frac{1}{f(z)-\omega}$ has (at most) a removable singularity at z_0.

$\Rightarrow \exists\, g$ analytic in $D \cup \{z_0\}$ s.t. $g(z) = \frac{1}{f(z)-\omega}$ $(\not\equiv 0)$

$\Rightarrow f(z) = \omega + \frac{1}{g(z)} = (\omega g(z) + 1)/g(z)$

$\Rightarrow f$ has either a pole (if $g(z_0)=0$) or a removable singularity (if $g(z_0) \neq 0$) at z_0 ($\longleftrightarrow\!\!\!\!\!\!\times$) #

Remark

In fact, there is a stronger thm (Great Picard's Thm): $\overset{f(D)}{R} = \mathbb{C}$ or $\mathbb{C} \setminus \{\text{a point}\}$

(Little Picard's Thm: if $f: \mathbb{C} \to \mathbb{C}$ is entire, nonconstant, then $f(\mathbb{C}) = \mathbb{C}$ or $\mathbb{C} \setminus \{\text{a point}\}$)

Laurent expansions

Def 9.7

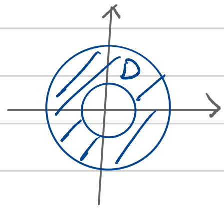
We say $\sum_{k=-\infty}^{\infty} u_k = L$ if both $\sum_{k=0}^{\infty} u_k$ and $\sum_{k=1}^{\infty} u_{-k}$ converge and if $\sum_{k=0}^{\infty} u_k + \sum_{k=1}^{\infty} u_{-k} = L$

We say $\sum_{k=-\infty}^{\infty} u_k$ is convergent if " " converge.
(resp. converges uniformly) (resp. converges uniformly)

Thm 9.8

$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ is convergent and analytic in the domain

$$D = \{z \in \mathbb{C} : R_1 < |z| < R_2\}$$



where $R_1 = \overline{\lim_{k\to\infty}} |a_{-k}|^{1/k}$, $\quad R_2 = 1/\overline{\lim_{k\to\infty}} |a_k|^{1/k}$

(Note if $R_1 \geq R_2$, then $D = \phi$)

pf

series	domain of convergence
$f_1(z) = \sum_{k=0}^{\infty} a_k z^k$	$\lvert z\rvert < 1/\overline{\lim_{k\to\infty}} \lvert a_k\rvert^{1/k} = R_2$
$f_2(z) = \sum_{k=-\infty}^{-1} a_k z^k = \sum_{k=1}^{\infty} a_{-k} z^{-k}$, $\quad z^{-k} = \left(\frac{1}{z}\right)^k$	$\frac{1}{\lvert z\rvert} < 1/\overline{\lim_{k\to\infty}} \lvert a_{-k}\rvert^{1/k} \iff \lvert z\rvert > \overline{\lim_{k\to\infty}} \lvert a_{-k}\rvert^{1/k} = R_1$

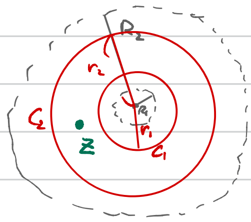
So, by Thm 2.9, $f(z) = f_1(z) + f_2(z)$ is analytic in D #

Thm 9.9

If f is analytic in the annulus $A = \{z \in \mathbb{C} : R_1 < |z| < R_2\}$, then f has a Laurent expansion, i.e. $\exists\ a_k\ \mathbb{C}$ s.t.

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k \qquad \forall z \in A$$

pf



① Let $C_j : r_j e^{i\theta}$, $\theta \in [0, 2\pi]$, $j = 1, 2$, $\quad R_1 < r_1 < r_2 < R_2$.

Fix z with $r_1 < |z| < r_2$. Then

$$g(w) = \frac{f(w) - f(z)}{w - z}$$

is analytic in A, and by Homotopy Thm (also see Example 2, p113),

$$\int_{C_2 - C_1} g(w)\, dw = 0$$

$$\Rightarrow \int_{C_2 - C_1} \frac{f(w)}{w - z}\, dw = \int_{C_2 - C_1} \frac{f(z)}{w - z}\, dw$$

Note that

$$\int_{C_2} \frac{1}{w - z}\, dw = 2\pi i \quad (\text{Lemma 5.4}) \qquad \int_{C_1} \frac{1}{w - z}\, dw = 0$$

$$\Rightarrow \int_{C_2 - C_1} \frac{f(z)}{w - z}\, dw = 2\pi i f(z)$$

$$\Rightarrow \boxed{f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{w - z}\, dw - \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w - z}\, dw} \quad (*)$$

cf. Cauchy Integral Formula, p61

② (a) For $w \in C_2$, since $|w| > |z|$

$$\frac{1}{w - z} = \frac{1}{w\left(1 - \frac{z}{w}\right)} = \frac{1}{w}\left(\sum_{k=0}^{\infty} \left(\frac{z}{w}\right)^k\right)$$

converges uniformly on C_2.

(b) For $w \in C_1$, since $|w| < |z|$,

$$\frac{1}{w - z} = -\frac{1}{z\left(1 - \frac{w}{z}\right)} = -\frac{1}{z}\left(\sum_{k=0}^{\infty} \left(\frac{w}{z}\right)^k\right)$$

converges uniformly on C_1.

③ Plug (a), (b) into (*):

$$f(z) = \frac{1}{2\pi i} \int_{C_2} \sum_{k=0}^{\infty} \frac{f(w)}{w^{k+1}} z^k\, dw + \frac{1}{2\pi i} \int_{C_1} \underbrace{\sum_{k=0}^{\infty} f(w) \cdot w^k \cdot \left(\frac{1}{z}\right)^{k+1}}_{= \sum_{k=-1}^{-\infty} \frac{f(w)}{w^{k+1}} z^k} dw$$

pf

By uniform convergence, we can switch $\int$ and Σ

$$\Rightarrow \quad f(z) = \sum_{k=-\infty}^{\infty} a_k z^k, \qquad a_k = \frac{1}{2\pi i}\int_{C_j} \frac{f(w)}{w^{k+1}}\,dw, \qquad j = \begin{cases} 2 & \text{if } k \geq 0 \\ 1 & \text{if } k < 0 \end{cases}$$

Note: since $\frac{f(w)}{w^{k+1}}$ is analytic in A, by Homotopy Thm,

$$a_k = \frac{1}{2\pi i}\int_C \frac{f(w)}{w^{k+1}}\,dw,$$

where C is any circle centered at 0 in A. $\Rightarrow$ a_k are independent of r_1, r_2, z (arbitrary in A) #

Prop (p.123)

The Laurent expansion is unique

pf

If

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

in $A = \{z \in \mathbb{C} : R_1 < |z| < R_2\}$

Let $C = \{re^{i\theta} : \theta \in [0, 2\pi]\}$, $r \in (R_1, R_2)$.

Since $\sum_{-\infty}^{\infty} a_n z^n$ converges uniformly along C, we have

exer: $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ converges uniformly on C

$$\int_C \frac{f(z)}{z^{k+1}}\,dz = \int_C \sum_{n=-\infty}^{\infty} a_n z^{n-k-1}\,dz = \sum_{n=-\infty}^{\infty} \int_C a_n z^{n-k-1}\,dz$$

Since

$$\int_C z^p\,dz = \int_0^{2\pi} r^p e^{ip\theta} \cdot r \cdot i e^{i\theta}\,d\theta = \begin{cases} 2\pi i & p = -1 \\ 0 & p \neq -1 \end{cases}$$

it follows that

$$\boxed{a_k = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{k+1}}\,dz}$$

#

Cor 9.10

If f is analytic in the annulus $R_1 < |z - z_0| < R_2$, then f has a unique representation

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

where

$$a_k = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{k+1}}\,dz$$

and $C = \{z_0 + re^{i\theta} : \theta \in [0, 2\pi]\}$, $R_1 < r < R_2$.

pf: Apply Thm 9.9 and Prop to $g(z) = f(z + z_0)$.

Cor 9.11

If f has an isolated singularity at z_0, then $\exists\, \delta > 0$ s.t.

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \qquad \forall\, 0 < |z - z_0| < \delta$$

where a_k are defined as in Cor 9.10.

pf: Apply Cor 9.10 to the case $R_1 = 0$.

Examples (p.124)

(i) $\frac{(z+1)^2}{z} = \frac{1}{z} + 2 + z \qquad \forall z \neq 0$

(ii) $\frac{1}{z^2(1-z)} = \frac{1}{z^2}(1 + z + z^2 + \cdots) = \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \cdots$ for $0 < |z| < 1$

(iii) $$\frac{1}{z^2(1-z)} = \frac{-1}{[1+(z-1)]^2 (z-1)} = \frac{+1}{z-1}\left(\frac{1}{1+(z-1)}\right)' = \frac{1}{z-1}\left(\sum_{n=0}^{\infty} (-(z-1))^n\right)'$$

$$= \frac{1}{z-1}\left(\sum_{n=0}^{\infty} (-1)^n n (z-1)^{n-1}\right) = \frac{-1}{z-1} + 2 - 3(z-1) + 4(z-1)^2 + \cdots \quad \text{for } 0 < |z-1| < 1$$

(iv) $\exp(1/z) = 1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots$ for $z \neq 0$

Laurent expansion and singularity

Def 9.12

If $f(z) = \sum a_k (z-z_0)^k$ is the Laurent expansion of f about an isolated singularity z_0, $\sum_{-\infty}^{-1} a_k (z-z_0)^k$ is called the principal part of f at z_0; $\sum_{0}^{\infty} a_k (z-z_0)^k$ is called the analytic part.

Remark (p124-125)

(i) If f has a removable singularity at z_0, all the coefficients a_{-k}, $k>0$, of its Laurent expansion about z_0 are 0.

pf

Assumption $\Rightarrow \exists$ analytic g s.t. $f(z) = g(z)$ $\forall D(z_0; \varepsilon) - \{z_0\}$, $\exists \varepsilon > 0$

Uniqueness of Laurent expansion $\Rightarrow$ Laurent expansion of f = Taylor expansion (of g) of g *

e.g. $\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$

(ii) If f has a pole of order k at z_0, then $a_{-k} \neq 0$ and $a_{-N} = 0$ $\forall N > k$

pf (p.125)

maybe skip

Assumption $\Rightarrow f(z) = \frac{A(z)}{B(z)}$ in a deleted nbd of z_0, A, B analytic at z_0, $A(z_0) \neq 0$, B has a zero of order k at z_0 ⊛

⊛ $\Rightarrow B(z) = \sum_{n=k}^{\infty} b_n (z-z_0)^n = (z-z_0)^k \cdot C(z)$, $C(z_0) \neq 0$, in a nbd of z_0

$\Rightarrow f(z) = \frac{Q(z)}{(z-z_0)^k}$ in a deleted nbd of z_0,

where $Q(z) = \frac{A(z)}{C(z)}$ is analytic and nonzero at z_0.

$\Rightarrow Q(z) = \sum_{n=0}^{\infty} C_n (z-z_0)^n$, $C_0 \neq 0$, near z_0

$\Rightarrow f(z) = \sum_{n=0}^{\infty} C_n \frac{(z-z_0)^n}{(z-z_0)^k} = \sum_{j=-k}^{\infty} a_j (z-z_0)^j$,

where $a_j = C_{j+k}$ $\Rightarrow a_{-k} = C_0 = Q(z_0) \neq 0$. #

(iii) If f has an essential singularity at z_0, it must have infinitely many nonzero terms in its principal part.

Cor 9.13 (Partial Fraction Decomposition of Rational Functions)

Suppose

$$R(z) = \frac{P(z)}{Q(z)} = \frac{P(z)}{(z-z_1)^{k_1}(z-z_2)^{k_2}\cdots(z-z_n)^{k_n}}$$

$z_1, \cdots, z_k$ are distinct

where $P(z)$ is a polynomial with $\deg P < \deg Q$, $\Rightarrow |R(z)| \to 0$ as $z \to \infty$, $Q(z) = (z-z_1)^{k_1} \cdots (z-z_n)^{k_n}$.

Then $R(z)$ can be expanded as a sum of polynomials in $1/(z-z_k)$, $k = 1, \cdots, n$.

pf

① Since R has a pole of order at most k_1 at z_1,

$$R(z) = P_1\left(\frac{1}{z-z_1}\right) + A_1(z)$$

where $P_1\left(\frac{1}{z-z_1}\right)$ is the principal part of R around z_1 and A_1 is the analytic part. analytic at $z_2, \cdots, z_n$

$\Rightarrow A_1(z) = R(z) - P_1\left(\frac{1}{z-z_1}\right)$ has a removable singularity at z_1 and the same principal parts as R at $z_2, \cdots, z_n$

② If we take $P_2\left(\frac{1}{z-z_2}\right)$ to be the principal part of R around z_2 and proceed inductively, we find

$$A_n(z) = R(z) - \left[P_1\left(\frac{1}{z-z_1}\right) + P_2\left(\frac{1}{z-z_2}\right) + \cdots + P_n\left(\frac{1}{z-z_n}\right)\right]$$

"is" an entire function. Furthermore, since $A_n \to 0$ as $z \to \infty$, $A_n(z)$ is bounded

By Liouville Thm (Thm 5.10), A_n = constant $= 0$ because $\deg P < \deg Q$

$\Rightarrow R(z) = P_1\left(\frac{1}{z-z_1}\right) + P_2\left(\frac{1}{z-z_2}\right) + \cdots + P_n\left(\frac{1}{z-z_n}\right)$ #