Ch 8 Simply connected domain

Simply connectedness

Def (homotopy)

Let $\gamma_0, \gamma_1 : [a,b] \to D$ be two piecewise C^1 curves in $D \subseteq \mathbb{C}$.

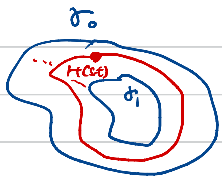
A homotopy between closed piecewise C^1 curves γ_0 and γ_1 is a continuous map $H: [a,b] \times [0,1] \longrightarrow D$ s.t.

① $\forall s_0 \in [a,b]$, the map $[0,1] \longrightarrow D : t \mapsto H(s_0, t)$ is a piecewise C^1 curve

② $\forall t_0 \in [0,1]$ the map $[a,b] \longrightarrow D : s \mapsto H(s, t_0)$ is a piecewise C^1 curve

③ $H(s,0) = \gamma_0(s)$, $H(s,1) = \gamma_1(s)$ $\forall s \in [a,b]$

④ $H(0,t) = H(1,t)$ $\forall t \in [0,1]$



We say γ_0 is homotopic to γ_1 if $\exists$ such H.

Def (Def 8.1)

An open connected set $D \subseteq \mathbb{C}$ is called simply connected if every closed piecewise C^1 curve in D is homotopic to a constant curve.

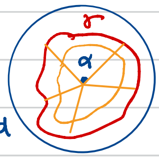
Remark

① "Simply connected" can be defined for any topological spaces ($\Leftarrow$ connected + $\pi_1 = 0$)
Def 8.1 in textbook only works for subsets in $\mathbb{C}$.

② In pictures, "simply connected" = connected + "no (2-dim) holes"

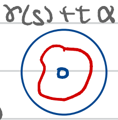
Example

(exer 1.2, Ch8)
exer: star-like $\Rightarrow$ simply connected



① For $r \in (0, \infty]$, $\alpha \in \mathbb{C}$, $D(\alpha; r)$ is simply connected
pf: $\forall$ closed piecewise C^1 curve $\gamma : [a,b] \to D(\alpha; r)$, $H(s,t) = (1-t)\gamma(s) + t\alpha$ is a homotopy

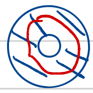
② $D(0;1) - \{0\}$ is NOT simply connected



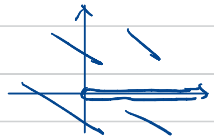
γ cannot pass through 0

③ $D(0;1) \cup D(3;1)$ is NOT simply connected (NOT connected)

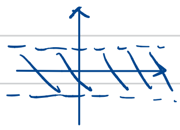
④ The annulus $A = \{ z \in \mathbb{C} : 1 < |z| < 3 \}$ is NOT simply connected



⑤ $\mathbb{C} \setminus \{ z \in \mathbb{C} : \mathrm{Re}(z) \geq 0, \mathrm{Im}(z) = 0 \}$ is simply connected



⑥ The strip $S = \{ z \in \mathbb{C} : -1 < \mathrm{Im}\, z < 1 \}$ is simply connected



General closed curve thm

This section: this can be replaced by any simply connected region

Recall (Closed curve thm Thm 6.3)

Suppose $r \in (0, \infty]$. If $f : D(z_0; r) \to \mathbb{C}$ is analytic, then $\forall$ closed piecewise C^1 curve C in $D(z_0; r)$,
$$\int_C f(z)\, dz = 0.$$

Thm (Homotopy Thm)

If γ_0 and γ_1 are two homotopic closed piecewise C^1 curves in a region D, then
$$\int_{\gamma_0} f(z)\, dz = \int_{\gamma_1} f(z)\, dz$$
for any analytic function $f : D \to \mathbb{C}$.

idea of pf

Divide $[a,b] \times [0,1]$ so that the image of each piece under H is in an open disc $D(z_j; \varepsilon)$.
By Thm 6.3, $\int_{C_j} f(z)\, dz = 0$ $\forall j$
$$\Rightarrow \int_{\gamma_0} f(z)\, dz - \int_{\gamma_1} f(z)\, dz = \sum_j \int_{C_j} f(z)\, dz = 0$$
#

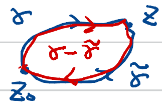
Cor (Thm 8.5, General Integral Thm)

Let D be a simply connected domain and $f: D \to \mathbb{C}$ be analytic. There exists an analytic function $F: D \longrightarrow \mathbb{C}$ s.t.

$$F'(z) = f(z) \qquad \forall z \in D$$

pf

Fix $z_0 \in D$. Since D is open connected, $\forall z \in D$, $\overset{\text{from advanced calculus}}{\exists}$ a piecewise C^1 curve $\gamma: [0,1] \to D$ s.t. $\gamma(0) = z_0$, $\gamma(1) = z$. Define



$$F(z) := \int_\gamma f(w)\, dw$$

① F is well-defined: Let $\tilde{\gamma}$ be another piecewise C^1 curve s.t. $\tilde{\gamma}(0) = z_0$, $\tilde{\gamma}(1) = 1$

Since D is simply connected, the closed curve $\gamma - \tilde{\gamma}$ is homotopic to a constant curve γ_0

$$\Rightarrow \int_\gamma f(w)\, dw - \int_{\tilde{\gamma}} f(w)\, dw = \int_{\gamma - \tilde{\gamma}} f(w)\, dw \overset{\text{Homotopy Thm}}{=} \int_{\gamma_0} f(w)\, dw = \int_a^b f(\gamma_0(t))\, \underbrace{\gamma_0'(t)}_{=0}\, dt = 0$$

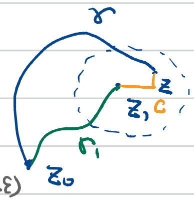
So $F(z)$ is independent of the choice of γ.

② Given any $z_1 \in D$, $\exists\, \varepsilon > 0$ s.t. $D(z_1; \varepsilon) \subseteq D$

Recall that, in Integral Thm (Thm 6.2), we consider $\tilde{F}: D(z_1; \varepsilon) \longrightarrow \mathbb{C}$ by

$$\tilde{F}(z) = \int_C f(w)\, dw$$

where C is the curve $z_1 \longrightarrow z_1 + \operatorname{Re}(z - z_1) \longrightarrow z$



By the proof of Integral Thm, $\tilde{F}'(z) = f(z)\ \forall z \in D(z_1; \varepsilon)$

③ Let $\gamma_1: [0,1] \to D$ be a curve s.t. $\gamma_1(0) = z_0$, $\gamma_1(1) = z_1$. By ①,

$$F(z) = \int_{\gamma_1} f(w)\, dw + \int_C f(w)\, dw = \underbrace{\int_{\gamma_1} f(w)\, dw}_{\text{fixed, indep of } z} + \tilde{F}(z) \qquad \forall z \in D(z_1; \varepsilon)$$

$$\Rightarrow F'(z) = \tilde{F}'(z) = f(z) \qquad \forall z \in D(z_1; \varepsilon)$$

Since z_1 is arbitrary in D, we have $F'(z) = f(z)$ $\forall z \in D$. #

We will denote $F(z) = \int_{z_0}^{z} f(w)\, dw$

Cor (Thm 8.6, General Closed Curve Thm)

Suppose f is analytic in a simply connected domain D and C is a closed piecewise C^1 curve in D. Then $\int_C f(z)\, dz = 0$

pf

Since D is simply connected, C is homotopic to a constant curve γ_0

$$\Rightarrow \int_C f(z)\, dz = \int_{\gamma_0} f(z)\, dz = 0$$

#

Example (p. 113)

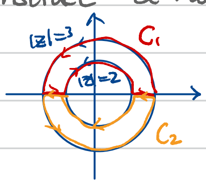
Suppose f is analytic in the annulus $A = \{ z \in \mathbb{C} : 1 < |z| < 4 \}$. Then

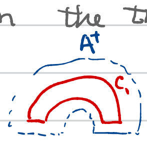
$$\int_{|z|=2} f(z)\, dz = \int_{|z|=3} f(z)\, dz$$

pf

method I: Construct a homotopy between the two circles $|z| = 2$ and $|z| = 3$.

method II:





Note that

$$\int_{|z|=3} f(z)\, dz - \int_{|z|=2} f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz \overset{\Uparrow\ \text{Thm 8.6}}{=} 0$$

and C_1 is contained in $A^+ = \{ z \in \mathbb{C} : 1 < |z| < 4,\ \operatorname{Im}(z) > -\frac{1}{2} \}$

C_2 is contained in $A^- = \{ z \in \mathbb{C} : 1 < |z| < 4,\ \operatorname{Im}(z) < \frac{1}{2} \}$ ← simply connected

#

Log

Def 8.7

We say f is an analytic branch of $\log z$ in a domain D if

(1) f is analytic in D

(2) $\exp(f(z)) = z \quad \forall z \in D$

Note $e^z \neq 0 \ \forall z \in \mathbb{C} \Rightarrow \log 0$ CANNOT be defined.

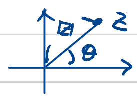
Remark

Let f be an analytic branch of $\log z$

① $g(z) = f(z) + 2\pi k i$ is also an analytic branch of $\log z$ for any fixed $k \in \mathbb{Z}$.

② Suppose $u(z) = \mathrm{Re}(f(z))$, $v(z) = \mathrm{Im}(f(z))$

$\mathrm{Arg}\, z = \theta + 2k\pi$



$$\Rightarrow z = \exp(u(z) + i v(z)) = e^{u(z)}(\cos v(z) + i \sin v(z)) = |z|(\cos\theta + i\sin\theta)$$

$$\Rightarrow e^{u(z)} = |z| \quad \text{and} \quad v(z) = \mathrm{Arg}\, z = \theta + 2k\pi$$

$$\Rightarrow \boxed{f(z) = \log|z| + i\,\mathrm{Arg}\, z}$$

Note: $\mathrm{Arg}\, z$ is NOT a well-defined function

Thm 8.8

Suppose D is a simply connected domain and $0 \notin D$.

Choose $z_0 \in D$, fix a value of $\log z_0$ and set

$$f(z) = \int_{z_0}^{z} \frac{1}{w}\, dw + \log z_0$$

Then $f(z)$ is an analytic branch of $\log z$ in D.

pf

f is well-def and analytic because $\frac{1}{w}$ is analytic in $D \not\ni 0$ and any paths from z_0 to z in D yield the same value (see pf of General Integral Thm = Thm 8.5)

Let

$$g(z) = z e^{-f(z)}$$

$$\Rightarrow g'(z) = e^{-f(z)} + z \cdot e^{-f(z)} \cdot (-\underbrace{f'(z)}_{\frac{1}{z}}) = e^{-f(z)} - e^{-f(z)} = 0 \quad \forall z \in D$$

$$\Rightarrow g(z) = \text{Constant} = g(z_0) = z_0 \cdot e^{-\log z_0} = 1$$

So $\quad z e^{-f(z)} = 1 \Rightarrow e^{f(z)} = z \quad \forall z \in D$ #

Remark

$$\left(\exp\left(\tfrac{1}{2}\log z\right)\right)^2 = \exp(\log z) = z$$

So " $\sqrt{z} = \exp\left(\tfrac{1}{2}\log z\right)$ "

More precisely, if f is an analytic branch of $\log z$, then $\exp(\frac{1}{2}f(z))$ is an analytic function s.t.

$$\left(\exp\left(\tfrac{1}{2}f(z)\right)\right)^2 = z$$

Similarly, $\left(\exp\left(\tfrac{1}{n}f(z)\right)\right)^n = z$

Note: Similar as $\log z$, $\sqrt{z}$ CANNOT be defined on arbitrary domain $D \subseteq \mathbb{C}$ analytically

Ch9 Isolated singularities of an analytic function

3 types of isolated singularities

Def 9.1 & 9.2

$D(z_0;\varepsilon)\sim\{z_0\}$ $\parallel$

A deleted neighborhood of $z_0 \in \mathbb{C}$ is a set of the form $D'(z_0;\varepsilon) = \{z\in\mathbb{C} : 0<|z-z_0|<\varepsilon\}$

A function f is said to have an isolated singularity at z_0 if f is analytic in a deleted nbd of z_0 but is NOT analytic at z_0

Suppose f has an isolated singularity at z_0

(i) If $\exists$ function g s.t. (a) g is analytic at z_0 (b) $f \equiv g$ in some deleted nbd of z_0, then we say f has a removable singularity at z_0

(ii) If $\exists$ A, B s.t. (a) A, B are analytic at z_0 (b) $A(z_0)\neq 0$, $B(z_0)=0$ (c) $f \equiv \frac{A}{B}$ in some deleted nbd of z_0, then we say f has a pole at z_0

If B has a zero of order k at z_0 (i.e. $B(z_0)=B'(z_0)=\cdots=B^{(k-1)}(z_0)=0$, $B^{(k)}(z_0)\neq 0$), then we say f has a pole of order k at z_0.

(iii) If f has neither a removable singularity nor a pole at z_0, then we say f has an essential singularity at z_0

Example (p.117)

(i) $f(z) := \begin{cases} \sin z, & z\neq 2 \\ 0, & z=2 \end{cases}$ has a removable singularity at $z=2$

(ii) $g(z) = \frac{1}{z-3}$ has a pole of order 1 at $z=3$

(iii) $\exp(\frac{1}{z})$ has an essential singularity at $z=0$

Remark

By Thm 7.7, if f is analytic in $D'(z_0;\varepsilon)$ ($\varepsilon>0$) and continuous at z_0, then f is analytic in $D(z_0;\varepsilon)$

So if f has an isolated singularity at z_0, then f must be discontinuous at z_0.

Removable singularity

Thm 9.3 (Riemann's Principle of Removable Singularities)

If f has an isolated singularity at z_0 and if $\lim_{z\to z_0}(z-z_0)f(z) = 0$, then the singularity is removable.

pf

Consider

$$h(z) = \begin{cases} (z-z_0)f(z) & z\neq z_0 \\ 0 & z=z_0 \end{cases}$$

By hypothesis, h is continuous at z_0. Since h, like f, is analytic in a deleted nbd of z_0, it follows that h is analytic at z_0 (continuous, analytic except one point, Thm 7.7)

Since $h(z_0)=0$,

$$g(z) = \begin{cases} \frac{h(z)}{z-z_0} & \text{for } z\neq z_0 \\ h'(z_0) & z=z_0 \end{cases}$$

is analytic at z_0 and $g(z)=f(z)$ $\forall z\neq z_0$ #

Cor 9.4

If f is bounded in a deleted nbd of an isolated singularity, the singularity is removable.

(pf: Assume z_0 is the removable singularity. bdd $\Rightarrow$ $f(z)(z-z_0)\to 0$)