Ch1 Complex numbers

Υ Complex numbers were borned from solving $x^2+1=0$
No sol in $\mathbb{R}$.
Extend $\mathbb{R}$

Field of complex numbers (viewpoint from algebra, §1.1)

Def 1.1

The field of complex numbers $\mathbb{C}$ is the set $\mathbb{R}\times\mathbb{R}$ together with the operations

$$(a,b)+(c,d) = (a+c,\ b+d)$$
$$(a,b)\cdot(c,d) = (ac-bd,\ ad+bc)$$

Prop

$(\mathbb{C},+,\cdot)$ is a field, i.e. $\forall\ x,y,z\in\mathbb{C}$,

- $x+(y+z) = (x+y)+z$, $(x\cdot y)\cdot z = x\cdot(y\cdot z)$ • $x+y=y+x$, $x\cdot y = y\cdot x$
- For $0=(0,0)$, $1=(1,0)\in\mathbb{C}$, $x+0=x$, $x\cdot 1 = x$ • for $x=(a,b)$, $x+(-a,-b)=0$
- for $x=(a,b)\neq(0,0)$, $\exists x^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)\in\mathbb{C}$ s.t. $x\cdot x^{-1} = (1,0) = 1$ ← pf : exer
- $x\cdot(y+z) = x\cdot y + x\cdot z$

Thm (will be proved in Thm5.12, p66)
Every nonconstant poly eg
$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$, $n>0$, $a_i\in\mathbb{C}$
has a sol in $\mathbb{C}$

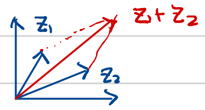
Remark

$(\mathbb{R},+,\cdot) \hookrightarrow (\mathbb{C},+,\cdot)$: $a\mapsto(a,0)$ is an embedding of fields.

Notation

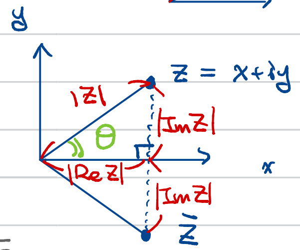
$(a,b) = a+ib = a+\sqrt{-1}\cdot b$ (Note: $(0,1)\cdot(0,1) = -1$)

Complex plane (viewpoint from linear algebra / plane geometry, §1.2)



$(\mathbb{C},+)$ is a 2-dim vector space over $\mathbb{R}$

Def



For $z = x+iy\in\mathbb{C}$,

$\mathrm{Re}\,z$ = real part of z $= x$

$\mathrm{Im}\,z$ = imaginary part of z $= y$

$\bar{z}$ = conjugate of z $= x-iy$

$|z|$ = absolute value of z = modulus of z $= \sqrt{x^2+y^2}$

$\mathrm{Arg}\,z$ = argument of z $=\theta$ s.t. $\cos\theta = \frac{\mathrm{Re}\,z}{|z|}$, $\sin\theta = \frac{\mathrm{Im}\,z}{|z|}$ (defined up to 2π)

Remark

For $z_1, z_2\in\mathbb{C}$, let $r_j = |z_j|$, $\theta_j = \mathrm{Arg}\,z_j$. Then $z_j = r_j(\cos\theta_j + i\sin\theta_j)$, and

$z_1 z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$ (in particular, $|z_1 z_2| = |z_1|\cdot|z_2|$)

$z_1^{-1} = \frac{1}{r_1}(\cos(-\theta_1) + i\sin(-\theta_1)) = r_1^{-1}(\cos\theta_1 - i\sin\theta_1)$ ← assume $z_1\neq 0$

$\frac{z_2}{z_1} = \frac{r_2}{r_1}(\cos(\theta_2-\theta_1) + i\sin(\theta_2-\theta_1))$

$z_1^n = r_1^n(\cos(n\theta_1) + i\sin(n\theta_1))$ $\forall n\in\mathbb{Z}$ ← if $n<0$

Topological aspects of complex plane (viewpoint from advanced calculus, §1.4)

$\mathbb{C}$ together with $d(z_1,z_2) = |z_1-z_2|$ is a metric space (in fact, $(\mathbb{C},|\cdot|)$ is a normed vector space)

skip so theorems in advanced calculus apply HW: read §1.4 and review relevant concepts

Def

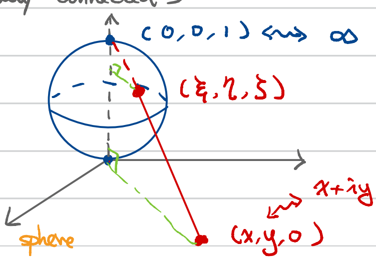
polygonal line = a finite union of line segments

A set $S\subseteq\mathbb{C}$ is polygonally connected if any 2 points in S can be connected by a polygonal line

An open connected set will be called a region ← Def 1.6 (Prop 1.7 A region is polygonally connected)

We say $\{z_k\}\to\infty$ if $|z_k|\to\infty$, $f(z)\to\infty$ if $|f(z)|\to\infty$ ← Def 1.11

Stereographic projection (§1.5)



HW: Prop 1.12, exer 25

By similar triangles, $\frac{x}{\xi} = \frac{y}{\eta} = \frac{1}{1-\zeta}$

$\Rightarrow$
- $x = \frac{\xi}{1-\zeta}$, $y = \frac{\eta}{1-\zeta}$
- $\xi = \frac{x}{x^2+y^2+1}$, $\eta = \frac{y}{x^2+y^2+1}$, $\zeta = \frac{x^2+y^2}{x^2+y^2+1}$

Remark

$\{z_k\}\to\infty$

$\leftrightsquigarrow$ $\{z_k\}\to(0,0,1)$ on sphere

Ch 2 - 3 Analytic functions

Analyticity and Cauchy-Riemann equation (§3.1)

Def 2.4

Let $f: U \xrightarrow{\text{open } \subset \mathbb{C}} \mathbb{C}$, $z \in U$. We say f is (complex) differentiable at z if the limit
$$\lim_{\substack{h\to 0\\ h\in\mathbb{C}}} \frac{f(z+h)-f(z)}{h}$$
exists. In this case, the limit is denoted $f'(z)$.

Remark

The limit $h\to 0$ is taken from all the possible directions in $\mathbb{C}$. So the condition "differentiable as complex function" is, in fact, much stronger than "differentiable as function of real variables".

Example: ① $f = \text{const}$ ② $f(x+iy) = x$ is NOT differentiable ③ $f(z) = \bar{z}$ 〃

Since the limit is of a same form as what we did in calculus, we have

Prop 2.5

If f, g are differentiable at z, then so are
$$h_1 = f+g, \qquad h_2 = fg$$
and if $g(z) \neq 0$,
$$h_3 = \frac{f}{g}.$$

Prop (exer 3, Ch 3)

If $f: \mathbb{C}\to\mathbb{C}$ is differentiable at z and if $g: \mathbb{C}\to\mathbb{C}$ is 〃 $f(z)$, then $g\circ f$ is 〃 at z, and
$$(g\circ f)'(z) = g'(f(z))\cdot f'(z)$$

In the respective cases,
$$h_1'(z) = f'(z) + g'(z), \quad h_2'(z) = f'(z)g(z) + f(z)g'(z), \quad h_3'(z) = (f'(z)g(z) - f(z)g'(z))/(g(z))^2$$

pf: exer 6 in Ch 2

Prop 3.1 (Cauchy-Riemann eq.)

Suppose $u, v: U \xrightarrow{\subset\mathbb{C}} \mathbb{R}$ s.t. $f = u+iv$. If f is differentiable at z ($= x+iy$), then f_x ($= u_x + iv_x$) and f_y ($= u_y + iv_y$) exist and satisfy the Cauchy-Riemann equation
$$f_y = i f_x$$
or, equivalently,
$$\star \quad \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

pf

Since $\displaystyle\lim_{h\to 0,\, h\in\mathbb{C}} \frac{f(z+h)-f(z)}{h}$ exists, we have

① $\displaystyle\lim_{\substack{\xi\to 0\\ \xi\in\mathbb{R}}} \frac{f((x+\xi)+iy) - f(x+iy)}{\xi} = f_x(x+iy)$ exists, and equals to

② $\displaystyle\lim_{\substack{\eta\to 0\\ \eta\in\mathbb{R}}} \frac{f(x+iy+i\eta) - f(x+iy)}{i\eta} = \frac{1}{i} f_y(x+iy)$

So $f_y = u_y + iv_y = if_x = iu_x + i^2 v_x = -v_x + iu_x$

$\Leftrightarrow \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ #

Example

Show that $f(x+iy) = x^2+y^3+ixy+2xy^2$ is NOT differentiable at $(1,1)$: $u_x = 2x+2y^2$, $v_y = x$ *

Prop 3.2 (partial converse of Prop 3.1)

Suppose f_x and f_y exist in a neighborhood U of z. If f_x and f_y are continuous at z and $f_y = if_x$ at z, then f is differentiable at z.

pf

Let $f = u+iv$, $h = \xi + i\eta$, $u, v: \mathbb{C}\to\mathbb{R}$, $\xi,\eta\in\mathbb{R}$. Write $u(z) = u(x,y)$, $v(z) = v(x,y)$. We'll estimate
$$\frac{f(z+h)-f(z)}{h} = \frac{1}{h}(u(z+h)-u(z)) + \frac{i}{h}(v(z+h)-v(z))$$

pf

By Mean Value Thm (for real functions of a real variable),

$$\frac{u(z+h)-u(z)}{h} = \frac{u(x+\xi, y+\eta) - u(x,y)}{\xi+i\eta}$$

$$= \frac{u(x+\xi, y+\eta) - u(x+\xi, y)}{\xi+i\eta} + \frac{u(x+\xi, y) - u(x,y)}{\xi+i\eta}$$

MVT

$$\overset{①}{=} \frac{\eta}{\xi+i\eta} u_y(x+\xi, y+\theta_1\eta) + \frac{\xi}{\xi+i\eta} u_x(x+\theta_2\xi, y) \quad \text{for some } 0<\theta_1, \theta_2<1$$

Recall (MVT)
If $g:\mathbb{R}\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) then $\exists\, c\in(a,b)$ s.t. $g'(c) = \frac{g(b)-g(a)}{b-a}$

($g(t) = u(x+\xi, y+t\eta)$, $b=1, a=0$) ect

and similarly,

$$\frac{v(z+h)-v(z)}{h} \overset{②}{=} \frac{\eta}{\xi+i\eta} v_y(x+\xi, y+\theta_3\eta) + \frac{\xi}{\xi+i\eta} v_x(x+\theta_4\xi, y) \quad \text{for some } 0<\theta_3, \theta_4<1$$

Thus,

$$\frac{f(z+h)-f(z)}{h} - f_x(z) = \frac{f(z+h)-f(z)}{h} - \left(\frac{\xi}{\xi+i\eta} f_x(z) + \frac{i\eta}{\xi+i\eta} f_x(z)\right)$$

by C-R eq $f_y = i f_x$ $\;= \frac{\eta}{\xi+i\eta} f_y$

$$\overset{①+②}{=} \frac{\xi}{\xi+i\eta}\Big(u_x(x+\theta_2\xi, \eta) - u_x(x,y) + i\big(v_x(x+\theta_4\xi, y) - v_x(x,y)\big)\Big)$$
$$+ \frac{\eta}{\xi+i\eta}\Big(u_y(x+\xi, y+\theta_1\eta) - u_y(x,y) + i\big(v_y(x+\xi, y+\theta_3\eta) - v_y(x,y)\big)\Big)$$

Since f_x, f_y are continuous at z ($\Rightarrow$ so are u_x, u_y, v_x, v_y) and $\left|\frac{\xi}{\xi+i\eta}\right|, \left|\frac{\eta}{\xi+i\eta}\right| \le 1$, we have

$$\left|\frac{f(z+h)-f(z)}{h} - f_x(z)\right| \le |u_x(x+\theta_2\xi, \eta) - u_x(x,y)| + |v_x(x+\theta_4\xi, y) - v_x(x,y)|$$
$$+ |u_y(x+\xi, y+\theta_1\eta) - u_y(x,y)| + |v_y(x+\xi, y+\theta_3\eta) - v_y(x,y)| \longrightarrow 0 \quad \text{as } h=\xi+i\eta\to 0$$

So $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exists, $= f_x(z)$. #

Remark

The assumptions in Prop 3.2 are necessary: Let $f(z) = f(x,y) = \begin{cases} \frac{xy(x+iy)}{x^2+y^2} & z\neq 0 \\ 0 & z=0 \end{cases}$

$\Rightarrow i f_x(0,0) = 0 = f_y(0,0)$ but $f'(0)$ does NOT exist $\left(u_x = \frac{2xy(x^2+y^2) - x^2y\cdot 2x}{(x^2+y^2)^2} = \frac{2xy^3}{(x^2+y^2)^2} \text{ NOT continuous at } z=0\right)$

Complex differentiability v.s. differentiability

Let $f: U \subseteq \mathbb{C}\cong\mathbb{R}^2 \longrightarrow \mathbb{R}^2 \cong \mathbb{C}$. Recall that ① f is differentiable (as real vector-valued function) at $z_0 = (x_0, y_0) \leftrightarrow x_0+iy_0$ if $\exists$ linear map $Df(z_0): \mathbb{R}^2 \to \mathbb{R}^2$ s.t.

$\mathbb{R}^2 \cong \mathbb{C}$

$$\lim_{z\in\mathbb{R}^2,\ z\to z_0} \frac{\|f(z)-f(z_0) - Df(z_0)(z-z_0)\|}{\|z-z_0\|} = 0 \qquad (\|z\| = |z| \text{ for } z\in\mathbb{C})$$

② For $f=(u,v): U\to\mathbb{R}^2$, if all $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist and are continuous on U, then f is differentiable on U as a real vector-valued function

Prop

If $f = u+iv: \underset{z_0\in}{U} \subseteq \mathbb{C}\cong\mathbb{R}^2 \longrightarrow \mathbb{C}\cong\mathbb{R}^2$ is complex differentiable at z_0, then f is differentiable at z_0 as a real vector-valued function and

$$Df(z_0)(z-z_0) = f'(z_0)\cdot(z-z_0)$$

$$\left(Df(z_0) = \begin{pmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{pmatrix} = \begin{pmatrix} a_0 & -b_0 \\ b_0 & a_0 \end{pmatrix} \text{ if } f'(z_0) = a_0+ib_0\right)$$

pf

By assumption, $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0) = f'(z_0)\cdot\frac{z-z_0}{z-z_0}$ $\forall\, z\neq z_0$

$$\Rightarrow \lim_{z\to z_0} \frac{|f(z)-f(z_0) - f'(z_0)\cdot(z-z_0)|}{|z-z_0|} = \left|\lim_{z\to z_0} \frac{f(z)-f(z_0) - f'(z_0)\cdot(z-z_0)}{z-z_0}\right| = 0$$

#

Example (converse is NOT true)

$f(z) = \bar{z} \quad \longleftrightarrow \quad f(x,y) = (\underset{u}{x}, \underset{v}{-y})$

Since all the partial derivatives exist and are continuous, f is differentiable on $\mathbb{C}\cong\mathbb{R}^2$.

But f is NOT complex differentiable because $u_x = 1 \neq -1 = v_y$ (doesn't satisfy C-R eq)

Convention

In this course, differentiable = complex differentiable unless otherwise stated.