Complex Analysis 6/9

We are considering the eq.

$$\circledast \begin{cases} \Delta u = 0 \\ u|_{\partial U} = g \end{cases}$$

given

where $U = D(0;1)$ is the unit disc

Thm 16.7 (uniqueness of the sol. of $\circledast$)

Suppose u is C-harmonic in U. Then

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\, K(\theta, z)\, dz \quad (z \in U)$$

where

$$K(\theta, z) := \mathrm{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)$$

is called the "Poisson kernel"

In polar form,

$$u(re^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{u(e^{i\theta})(1-r^2)}{1-2r\cos(\theta-\varphi)+r^2}\, d\theta$$

pf (sketch)

① Show that $u = \mathrm{Re} f$ for some f analytic in $\overline{U}$

② Recall that by Cauchy Integral Formula (Thm 6.4),

$$f(z) = \frac{1}{2\pi i}\int_{|w|=1} \frac{f(w)}{w-z}\,dw$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\left(\frac{e^{i\theta}}{e^{i\theta}-z}\right) d\theta \quad \text{— ⓐ}$$

③ For $z \in U$, $\dfrac{f(w)}{w - \frac{1}{\bar z}}$ is analytic $\forall\, w \in U$.

$\Rightarrow$

$$0 = \frac{1}{2\pi i}\int_{|w|=1} \frac{f(w)}{w-\frac{1}{\bar z}}\,dw$$

← Closed Curve Thm (Thm 8.6)

$$= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\left(\frac{e^{i\theta}}{e^{i\theta}-\frac{1}{\bar z}}\right) d\theta \quad \text{— ⓑ}$$

④ By computing $\mathrm{Re}(ⓐ - ⓑ)$, one can prove the theorem. #

Thm 16.8 (Dirichlet Problem, existence of sol of ⊛)

Suppose $g : \{|z|=1\} = \partial U \longrightarrow \mathbb{R}$ is continuous.

Then

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\, K(\theta, z)\, d\theta$$

is the solution of

$\{\, \Delta u = 0$ in U

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u(e^{i\theta}) = g(e^{i\theta}) \end{cases}$$

pf: skip. See p.230 – 231.

Remark

By considering appropriate conformal mapping, one can solve the Dirichlet Problem a bounded simply connected domain

$$\begin{cases} \Delta u = 0 \\ u|_{\partial D} = g \end{cases} \quad \text{in } D \underset{\phi^{-1}}{\overset{\phi}{\rightleftarrows}} U$$

conformal maps

Solve $(\tilde{u} = u \circ \phi^{-1})$

$$\begin{cases} \Delta \tilde{u} = 0 \\ \tilde{u}|_{\partial U} = g \circ \phi^{-1} \end{cases}$$

Then $u = \tilde{u} \circ \phi$ is a sol

Example

Solve
$$\begin{cases} \Delta u = 0 & \text{in } U = D(0;1) \\ u(x,y) = x^2 & \forall (x,y) \in \partial U \end{cases}$$

sol

Note that

$$\mathrm{Re}(z^2) = x^2 - y^2 \qquad \Rightarrow y^2 = 1 - x^2$$

and if $|z|^2 = 1 = x^2 + y^2$, then

$$\mathrm{Re}(z^2) = 2x^2 - 1 \qquad \text{on } |z| = 1$$

$\Rightarrow$ if $|z| = 1$,

$$\mathrm{Re}\left((z^2+1)/2\right) = x^2 \qquad \text{on } |z|=1$$

$$\Rightarrow \quad u(x,y) = \mathrm{Re}\left(\frac{z^2+1}{2}\right) = \frac{1}{2}(x^2-y^2) + \frac{1}{2}$$

is a sol. #

Heat equation

Let $u(x,y;t)$ be the temperature at the point (x,y), time t.

A physical law says u satisfies the heat eq:

$$\frac{\partial u}{\partial t} = \Delta u$$

When the distribution of heat doesn't change anymore ("steady-state"), i.e., if $\frac{\partial u}{\partial t} = 0$, then

$$\Delta u = 0$$

(no heat source steady state)

So, for example, the solution

$$u(x,y) = \frac{1}{2}(x^2 - y^2) + \frac{1}{2}$$

could be a function of temperature distribution on a unit disc.

National Tsing Hua University
Complex Analysis – Exam 4

Instructor: Hsuan-Yi Liao

Spring, 2022

Name: ______________________________

Student ID: ______________________________

- This exam contains 10 pages (including this cover page) and 9 questions.
- Total of points is 25.
- Write down your computation or arguments in details unless otherwise stated.
- In this exam, assume
  - $\mathfrak{s}$ = your student ID;
  - $\tilde{\mathfrak{s}} = 2 + \Big|(\text{the last digit of your student ID}) - 5\Big|$.

  For example, if your student ID is 66666, then $\mathfrak{s} = 66666$ and $\tilde{\mathfrak{s}} = 3$.
- Plug numbers into the parameters $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$ in your answers.

Distribution of Marks

Question	Points	Score
1	3	
2	3	
3	3	
4	3	
5	3	
6	3	
7	3	
8	2	
9	2	
Total:	25	

1. (3 points) Show that there are no analytic functions $f = u + iv$ with $u(x, y) = x^2 + y^2$.

Method I

The real part of an analytic function is harmonic, but $u(x,y) = x^2+y^2$ is not harmonic. $\Rightarrow$ o.k. #

$$\Delta u = u_{xx} + u_{yy} = 2 + 2 = 4 \neq 0$$

Method II

By C-R eq.,

$$\begin{cases} 2x = u_x = \overset{2}{v_y} \\ 2y = u_y = -v_x \end{cases} \Rightarrow \begin{cases} v = 2xy + g(x) \\ v = -2yx + h(y) \end{cases}$$

for some real-valued functions g, h

(v, $2xy$ are differentiable $\Rightarrow$ so are g and h)

$$\Rightarrow \underbrace{(2xy + g(x))_{xy}}_{=\,2} = \underbrace{(-2xy + h(y))_{xy}}_{=\,-2} \qquad (\rightarrow\leftarrow) \quad \#$$

2. (3 points) Prove that a nonconstant entire function *cannot* satisfy the two equations

   i. $f(z+1) = f(z)$

   ii. $f(z+i) = f(z)$

   for all z.

3. Find the Laurent expansion for

(a) (1 point) $\frac{1}{z^4 + z^2}$ about $z = 0$

(b) (2 points) $\frac{1}{z^2 - 4}$ about $z = 2$.

A wrong answer:

$$\frac{1}{z^2-4} = (z-2)^{-1}\frac{1}{z-2+4} = (z-2)^{-2}\frac{1}{1+\frac{4}{z-2}}$$

$$= (z-2)^{-2}\sum_{n=0}^{\infty}\left(-\frac{4}{z-2}\right)^n = \sum_{n=0}^{\infty}(-4)^n (z-2)^{-n-2}$$

$$\forall \left|\frac{4}{z-2}\right| < 1 \quad \text{i.e.} \quad |z-2| > 4$$

problem: NOT a deleted nbd $0 < |z-2| < \varepsilon$

$\Rightarrow$ NOT a Laurent expansion

A common mistake:

domain of convergence: $0 < |z-a| < 1$!!

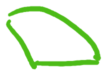
4. (3 points) Find the number of zeros (counting multiplicities) of $f(z) = z^4 - 5z + 1$ in $1 \leq |z| \leq 2$

5. (3 points) Evaluate the integral $\displaystyle\int_0^\infty \frac{1}{\sqrt[3]{x}(1+x)}\,dx$.

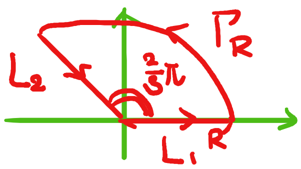
Calculus method: $u = \sqrt[3]{x} \Rightarrow u^3 = x \Rightarrow 3u^2 du = dx$

$$\int_0^\infty \frac{1}{\sqrt[3]{x}\,(1+x)}\,dx = \int_0^\infty \frac{1}{u(1+u^3)}\,3u^2\,du = \int_0^\infty \frac{3u}{1+u^3}\,du$$

$= \cdots$

or use  (Residue Thm)

Method II:

Let 

$C_R = L_1 \cup \Gamma_R \cup L_2$

By Residue Thm, for $R > 1$,

$$\int_{C_R} \frac{3u}{1+u^3}\,du = 2\pi i \operatorname{Res}\left(\frac{3u}{1+u^3}; e^{\frac{\pi}{3}i}\right)$$

$$= 2\pi i \left.\frac{\cancel{3}u}{\cancel{3}u^2}\right|_{u=e^{\frac{\pi}{3}i}}$$

$$= 2\pi i \cdot e^{-\frac{\pi}{3}i}$$

Note: $\displaystyle\left|\int_{\Gamma_R} \frac{3u}{1+u^3}\,du\right| \le \frac{2\pi}{3}R \cdot \frac{3R}{1+R^3} \longrightarrow 0$ as $R \to \infty$

$\left(L_2: (R-t)\cdot e^{\frac{2}{3}\pi i},\ t \in [0,R]\right)$ $\displaystyle\int_{L_2} \frac{3u}{1+u^3}\,du = -\int_0^R \frac{3\cdot t e^{\frac{2}{3}\pi i}}{1+\underbrace{(te^{\frac{2}{3}\pi i})^3}_{=t^3}}\,e^{\frac{2}{3}\pi i}\,dt$

$$= -e^{\frac{4}{3}\pi i}\int_0^R \frac{3t}{1+t^3}\,dt = -e^{\frac{4}{3}\pi i}\int_{L_1} \frac{u}{1+u^3}\,du$$

$$\Rightarrow 2\pi i \cdot e^{-\frac{\pi}{3}i} = \left(1 - e^{\frac{4}{3}\pi i}\right)\int_0^\infty \frac{3u}{1+u^3}\,du$$

$$\Rightarrow \text{ans} = \int_0^\infty \frac{3u}{1+u^3}\,du = \frac{2\pi i \cdot \cancel{e^{-\frac{\pi}{3}i}}}{\cancel{1 - e^{\frac{4}{3}\pi i}}\ e^{\frac{\pi}{3}i} - e^{\frac{5}{3}\pi i}} = \frac{2\pi i}{2i\sin\frac{\pi}{3}} = \frac{\pi}{\sin\frac{\pi}{3}}$$

$= e^{-\frac{\pi}{3}i}$ $= \frac{2\pi}{\sqrt{3}}$ #

6. (3 points) Show that

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}$$

$(1-x)^{\alpha}$

as long as $|x| < \frac{1}{4}$.

Calculus method: Compute Taylor expansion of $\frac{1}{\sqrt{1-4x}}$ directly

7. (3 points) Let R be a simply connected domain and assume $z_1, z_2 \in R$. Show that there exists a conformal mapping of R onto itself, taking z_1 to z_2. (Consider two cases: $R \neq \mathbb{C}$ and $R = \mathbb{C}$.)

8. (2 points) Evaluate the integral

$$\int_0^{2\pi} \frac{1}{1+\tilde{\mathfrak{s}}+\tilde{\mathfrak{s}}\sin x}\,dx.$$

Calculus method: $t = \tan\frac{x}{2} \Rightarrow \sin x = \frac{2t}{1+t^2}$ ...

9. (2 points) Let $R \subset \mathbb{C}$ be the open set

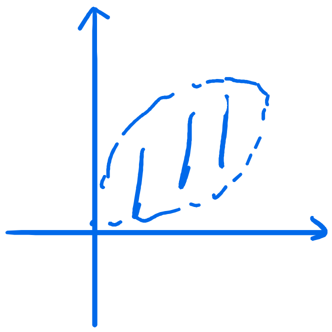
$$R = \{z \in \mathbb{C} : |z-1| < 1 \text{ and } |z-i| < 1\}.$$

Find a conformal mapping f from R onto the unit disk U with the property

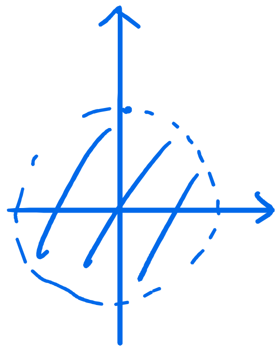
$$f(\frac{1}{\mathfrak{s}} + \frac{i}{\mathfrak{s}}) = 0.$$

(See HW11)

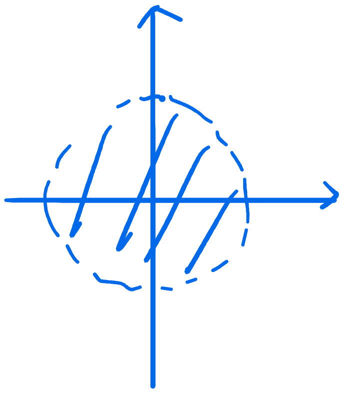
Step1: find conformal



$\xrightarrow{f}$



Step2: Compose



with $B_\alpha(z) = \dfrac{z-\alpha}{1-\bar{\alpha}z}$

where $\alpha = f(\frac{1}{s} + \frac{i}{s}) \in \{|z|<1\}$

$$\Rightarrow (B_\alpha \circ f)(\frac{1}{s} + \frac{i}{s})$$

$$= B_\alpha(\alpha) = 0$$

Summary of this course

- Basic properties of analytic functions (Ch2-3)
  - *Cauchy-Riemann eq
  - examples: polynomials, e^z, $\sin z$, $\cos z$, functions defined by power series, $\log z$ (§8.2)
- *Line integrals (Ch4, Ch6, Ch8)

  $C: z(t)$ ← eg $\int_C f(z)dz = \int_a^b f(z(t))\cdot z'(t)\,dt$
  - basic properties and M-L inequality: $\left|\int_C f(z)dz\right| \le \sup_{z\in C}|f(z)|\cdot \text{length}(C)$
  - closed curve thm & integral thm: domain is important!! (need simply connected)
- Applications of line integrals to analytic functions (Ch5-7)
  - Cauchy integral formula: $f(a) = \frac{1}{2\pi i}\int_{|z|=R}\frac{f(z)}{z-a}dz$ (more generally, residue thm)
  - power series of analytic functions, uniqueness of analytic function
  - Liouville thm, MVT, Max-Modulus, open mapping thm, Morera thm, reflection principle, etc.
  - Schwarz' lemma — applied to Riemann Mapping Thm (§14.2)
- Analytic functions with isolated singularities (§9.1 - §10.1)
  - classification of isolated singularities
  - Laurent expansion around an isolated singularity $\sum_{n=-\infty}^{\infty} a_n z^n$
  - residue: c_{-1} in Laurent expansion
  - *residue thm: $\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{m} n(\gamma, z_k)\cdot \text{Res}(f; z_k)$ ← winding number; isolated singularities inside the closed curve γ
- Applications of residue thm (§10.2 — Ch12)
  - argument principle, Rouché thm: count number of zeros
  - computation: integrals, sums (e.g. $\sum_{n=1}^{\infty}\frac{1}{n^2}$ ⇝ zeta function)
- Conformal mapping (Ch13-14)
  - definitions and basic properties
  - Riemann Mapping Thm: any proper (i.e. $\neq \emptyset$, $\neq \mathbb{C}$) simply connected domains are conformally equivalent
  - examples of conformal maps: $az+b$, z^a, e^z, bilinear transformations $\frac{az+b}{cz+d}$, $ad-bc\neq 0$
  - construction of conformal maps
- Harmonic functions (Ch16)
  - relationship between harmonic functions and analytic functions
  - $\exists!$ problems of the eq. $\begin{cases}\Delta u = 0 \\ u|_{\partial\Omega} = g \leftarrow \text{given}\end{cases}$ cf. heat eq.
    ($\Omega = D(0;1)$)