Complex Analysis 6/6

Ch16 Harmonic function

Def 16.1

$u : D \to \mathbb{R}$

($D \subset \mathbb{R}^2$ open)

A real-valued function $u(x,y)$ which is twice continuously differentiable and satisfies Laplace's equation

$$\Delta u := u_{xx} + u_{yy} = 0$$

throughout a domain D is said to be harmonic in D.

e.g. $u(x,y) = x^2 - y^2$ is harmonic in $\mathbb{R}^2$

because

$$u_{xx} + u_{yy} = (x^2 - y^2)_{xx} + (x^2 - y^2)_{yy}$$

$$= 2 + (-2) = 0$$

Note $z = x + iy$

$\mathrm{Re}(z^2) = x^2 - y^2$

Thm 16.2

If $f = u + iv$ is analytic in D, then u and v are harmonic in D

pf

Recall $f = u + iv$ is analytic $\Rightarrow$ f satisfies the Cauchy-Riemann eq. *

$f_y = i f_x \iff \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ in D.

$$\Rightarrow \quad u_{xx} = (v_y)_x = (v_x)_y = (-u_y)_y \quad \text{in } D$$

$$v_{xx} = (-u_y)_x = -(u_x)_y = -(v_y)_y$$

$$\Rightarrow \quad \Delta u = 0, \quad \Delta v = 0 \quad \text{in } D \qquad \#$$

<u>Remark</u>

The converse of <u>Thm 16.2</u> is NOT true.

<u>Example</u> (exer 4, Ch16)

The function

harmonic by computation

$$u(x,y) = \log(x^2+y^2)$$

is harmonic in $D = \mathbb{R}^2 - \{(0,0)\}$, but it is NOT the real part of an analytic function in D.

<u>pf</u>

$re^{i\theta}$, $r = \sqrt{x^2+y^2}$, $\Rightarrow \log z = \log r + i\theta$

Note that if $z = x+iy \neq 0$, then

$$\mathrm{Re}(\log z) = \log\sqrt{x^2+y^2}$$

$$= \frac{1}{2}\log(x^2+y^2) = \frac{1}{2}u(x,y)$$

Suppose $f: D \to \mathbb{C}$ is analytic s.t.

$$\mathrm{Re} f = u = \mathrm{Re}(2\log z)$$

Here, $\log z$ is the analytic branch in $\mathbb{C} \setminus \mathbb{R}_{\le 0}$ with $\log 1 = 0$

Then

$$f(z) - 2\log z$$

is analytic in $\mathbb{C} \setminus \mathbb{R}_{\le 0}$ with $\mathrm{Re}(f(z) - \log z) \equiv 0$

By Cauchy-Riemann eq.

$$\mathrm{Im}(f(z) - \log z) = \text{real constant } C$$

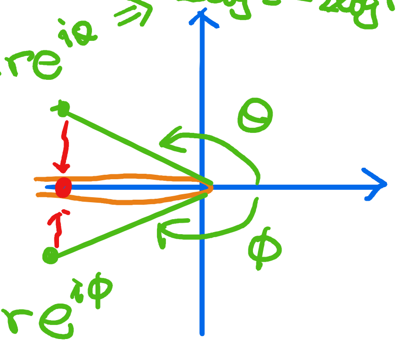
$$\Rightarrow \quad f(z) - 2\log z = iC \quad \text{in } \mathbb{C} \setminus \mathbb{R}_{\le 0}$$

However,

$$f(z) = 2\log z + iC$$

cannot be continuously extended to $D = \mathbb{C} \setminus \{0\}$ $(\rightarrow\leftarrow)$.

$z = re^{i\theta} \Rightarrow 2\log z = 2\log r + 2i\theta$



$$\mathrm{Im}\, f(re^{i\theta}) = 2\theta + C \sim 2\pi + C$$

$$\#$$

$$\mathrm{Im}\, f(re^{i\phi}) = 2\phi + C \sim -2\pi + C$$

#

Thm 16.3

If u is harmonic in D, then

a. u_x is the real part of an analytic function in D

b. if D is simply connected, then u is the real part of an analytic function in D

pf

a. Let $\boxed{f = u_x - i u_y}$

Since $u \in C^2(D)$, f has continuous first-order partial derivatives in D

Moreover,

$$f_y = u_{xy} - i \underbrace{u_{yy}}_{= -u_{xx}} = u_{yx} + i u_{xx}$$

C-R eq:

$$= i(u_x - i u_y)_x = i f_x \quad \text{in } D$$

By Prop 3.2, f is analytic in D

b. If D is simply connected, by Integral Thm (Thm 8.5), $\exists$ an analytic function $F = A + iB$ in D s.t.

$$F' = f = u_x - i u_y \leftarrow \text{analytic by a.}$$

$$\Rightarrow F'(z) = A_x + i \underbrace{B_x}_{= -A_y} \overset{C\text{-}R}{=} A_x - i A_y$$

$$= u_x - i u_y$$

$$\Rightarrow \begin{cases} A_x = u_x \\ A_y = u_y \end{cases} \quad \text{in } D$$

$\Rightarrow \quad A(x,y) = u(x,y) + C$ for some constant $C \in \mathbb{R}$

$\Rightarrow \quad u = A - C = \mathrm{Re}(F) - C = \mathrm{Re}(\underbrace{F-C}_{\text{analytic in } D})$ #

<u>Example</u>

$u(x,y) = x - e^x \sin y$ is harmonic in $\underbrace{\mathbb{R}^2}_{\text{simply connected}}$

because

$$\left.\begin{aligned} u_{xx} &= -e^x \sin y \\ u_{yy} &= +e^x \sin y \end{aligned}\right\} \Rightarrow \quad \Delta u = 0$$

Find an entire function F s.t. $\mathrm{Re}\,F = u$.

<u>sol</u>

By the proof of <u>Thm 16.3.a</u>, the function

$f(z) = u_x(x,y) - i\,u_y(x,y)$

$= (\underbrace{1 - e^x \sin y}) - \underbrace{i(-e^x \cos y)}$

is entire. $\qquad ie^z = i\,e^x(\cos y + i \sin y)$

In fact,

$$f(z) = 1 + ie^z$$

If we set

$$F(z) = \int_0^z f(w)\,dw = z + ie^{z} - i$$

then

$$\operatorname{Re} F(z) = x - e^{x}\sin y = u(x,y)$$

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Mean-Value Thm for harmonic functions

Thm 16.4 (MVT)

If u is harmonic in $D(z_0; R)$, then for $0 < r < R$,

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta$$

pf

Since $D(z_0; R)$ is simply connected, we may assume (Thm 16.3) $u = \operatorname{Re} f$ for some f analytic in $D(z_0; R)$

By Thm 6.12 (MVT for analytic function)

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta \quad (*)$$

By taking the real parts of $(*)$,

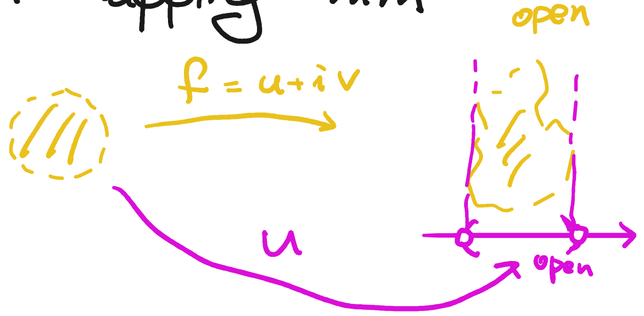
we can prove Thm 16.4 #

We can prove Thm16.4 . #

Thm16.5 (Maximum-Modulus Thm)

If u is a nonconstant harmonic function in a region $D \subset_{\text{open}} \mathbb{C} \cong \mathbb{R}^2$, then u has no maximum or minimum points in D

pf: By MVT or Open Mapping Thm



Def

We say a function is C-harmonic if it is continuous and is harmonic in the interior

Thm16.5 $\Rightarrow$ A C-harmonic function in a compact domain must assume its maximum and minimum values on the boundary of the domain

Cor16.6

If two C-harmonic functions u_1 and u_2 agree on the boundary of a compact domain D, then $u_1 \equiv u_2$ in D.

pf

$u := u_1 - u_2$ is C-harmonic in D, so

u takes its max and min on ∂D.

Since $u \equiv 0$ on ∂D, we have

$$0 \leq u \leq 0 \quad \text{in } D$$

$\Rightarrow \quad u_1 \equiv u_2$ in D #

Therefore, a C-harmonic function in a compact domain is determined by its values on the boundary of the domain

Next:

How can we determine the C-harmonic function explicitly by its values on the boundary?

Cf: "Dirichlet Problem"

Solve
$$\begin{cases} \Delta u = 0 \\ u|_{\partial D} = \tilde{u} \end{cases}$$
some given function