Complex Analysis 6/2

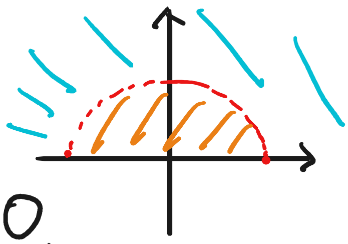
<u>Problem 10 in HW6</u>

Suppose f is analytic in the semi-disc:

$$\{|z|<1,\ \operatorname{Im} z>0\},$$

continuous on $\{|z|\le 1,\ \operatorname{Im} z>0\}$ and real the semi-circle $|z|=1$, $\operatorname{Im} z>0$.

Show that if we set



$$g(z)=\begin{cases} f(z), & |z|\le 1,\ \operatorname{Im} z>0 \\ \overline{f\left(\frac{1}{\bar z}\right)}, & |z|>1,\ \operatorname{Im} z>0, \end{cases}$$

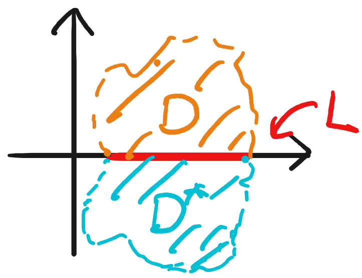
then g is analytic in $\{\operatorname{Im} z>0\}$

<u>pf</u>

<u>Recall</u> (Schwarz reflection principle, <u>Thm 7.8</u>)

Suppose f is analytic in a domain D and continuous in $\bar D$. Suppse ∂D contains a line segment L on the real axis, and

$$f(L)\subseteq \mathbb{R}.$$



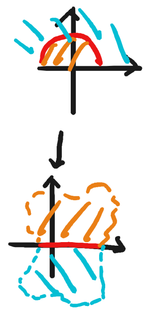
Then

$$g(z):=\begin{cases} f(z), & z\in D\cup L \\ \overline{f(\bar z)}, & z\in D^* \end{cases}$$

is analytic in $D \cup L \cup D^*$, where

$$D^* = \{z \in \mathbb{C} : \bar{z} \in D\}$$

<u>Idea</u>: transfer Schwarz reflection principle by a suitable conformal map

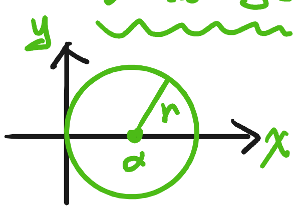


<u>Need</u>: a conformal map which maps the semi-circle, $|z|=1$, $\text{Im} z > 0$, to a line segment L on the real axis

<u>Construction of conformal map</u>

<u>Recall</u>

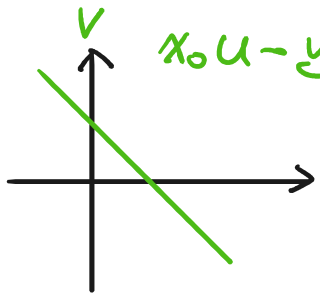
$\alpha = x_0 + iy_0$



$|z-\alpha| = r$

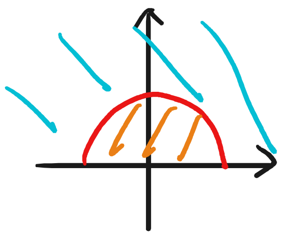
$r = |\alpha|$

$z \mapsto \frac{1}{z}$

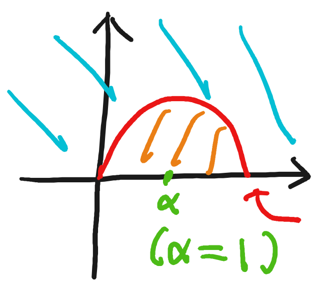


$x_0 u - y_0 v = \frac{1}{2}$

<u>Construction</u>



$z \mapsto z+1$



$\text{Im} z > 0$

$(\alpha = 1)$

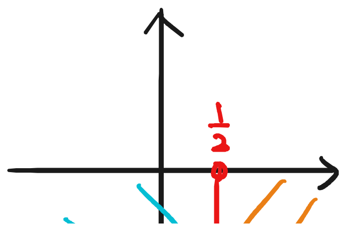
$x_0 = 1,\ y_0 = 0$

$x - 0y = \frac{1}{2}$ i.e. $x = \frac{1}{2}$

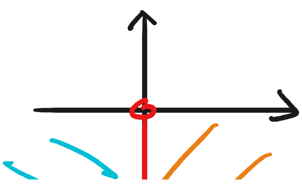
$z \mapsto \frac{1}{z}$

$z = x+iy$

$\text{Im} z > 0$



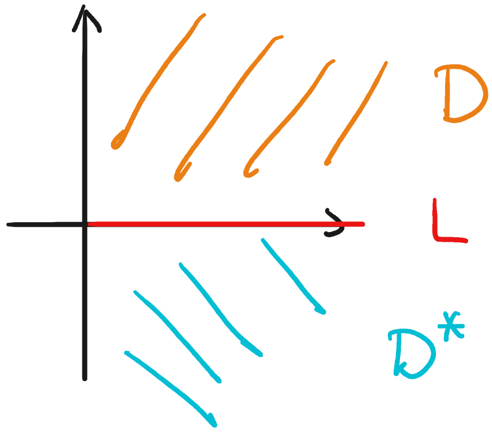
$z \mapsto z - \frac{1}{2}$



$$\Rightarrow \frac{1}{z} = \frac{x-iy}{x^2+y^2}$$

$$\operatorname{Im}\left(\frac{1}{z}\right) < 0 = \frac{1}{x+iy}$$

$$z \to e^{i\frac{\pi}{2}} z = iz$$

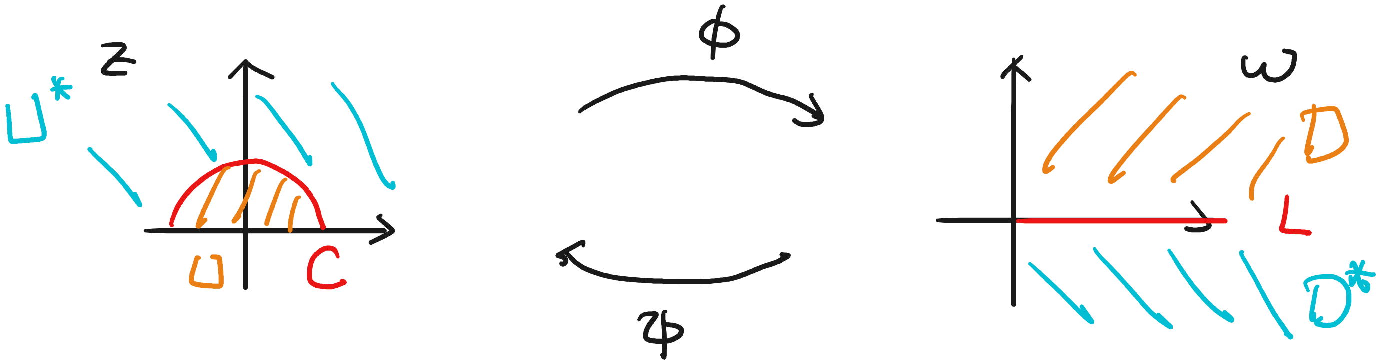


The whole transformation is

$$z \mapsto z+1 \longmapsto \frac{1}{z+1} \longmapsto \frac{1}{z+1} - \frac{1}{2} \mapsto i\left(\frac{1}{z+1} - \frac{1}{2}\right) = \frac{-iz+i}{2z+2}$$

<u>Transfer the principle</u>

Let $\Phi(z) = \dfrac{-iz+i}{2z+2}$, $\Psi(w) = \dfrac{2w-i}{-2w-i} = \dfrac{i-2w}{2w+i}$



Since f is analytic in U, continuous in $U \cup C$, real on C, we

$f \circ \psi$

is analytic in U, continuous in $D \cup L$, real on L

By Schwarz reflection principle,

$$h(\omega) := \begin{cases} f(\Psi(\omega)), & \omega \in D \cup L \\ \overline{f(\Psi(\bar{\omega}))}, & \omega \in D^* \end{cases}$$

is analytic in $D \cup L \cup D^*$

$\Rightarrow$ $g := h \circ \phi$ is analytic in $U \cup C \cup U^*$

<u>Formula of g</u>

$$g(z) = h(\Phi(z)) = \begin{cases} f(\underbrace{\Psi(\Phi(z))}_{}) \overset{\text{id}}{=} f(z), & z \in U \cup C \\ \overline{f(\underbrace{\Psi(\overline{\Phi(z)})}_{})}, & z \in U^* \end{cases}$$

and

$$\Psi(\overline{\Phi(z)}) = \Psi\left(\overline{\frac{i - iz}{2z+2}}\right) = \Psi\left(\frac{-i + i\bar{z}}{2\bar{z}+2}\right)$$

$$= \left(i - 2\left(\frac{-i + i\bar{z}}{2\bar{z}+2}\right)\right) \Big/ \left(2\left(\frac{-i + i\bar{z}}{2\bar{z}+2}\right) + i\right)$$

$$= \frac{1}{}$$

$\bar{z}$

So

$$g(z) = \begin{cases} f(z) & z \in U \cup C \\ \overline{f(1/\bar{z})} & z \in U^* \end{cases}$$

is analytic in $U \cup C \cup U^*$. #

<u>exer</u>: Try to get other forms of theorems.

National Tsing Hua University

Complex Analysis – Exam 4

Instructor: Hsuan-Yi Liao

Spring, 2022

Name: ______________________________

if

Student ID: 9521204 ______________________________

- This exam contains 10 pages (including this cover page) and 9 questions.
- Total of points is 25.
- Write down your computation or arguments in details unless otherwise stated.
- In this exam, assume
  - $\mathfrak{s}$ = your student ID; $\mathfrak{s} = 9521204$
  - $\tilde{\mathfrak{s}} = 2 + \Big|$(the last digit of your student ID) $- 5\Big|$. $\tilde{\mathfrak{s}} = 2 + |4-5| = 2+1 = 3$

  For example, if your student ID is 66666, then $\mathfrak{s} = 66666$ and $\tilde{\mathfrak{s}} = 3$.
- Plug numbers into the parameters $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$ in your answers.

Distribution of Marks

Question	Points	Score
1	3	
2	3	
3	3	
4	3	
5	3	
6	3	
7	3	
8	2	
9	2	
Total:	25	

1. (3 points) Show that there are no analytic functions $f = u + iv$ with $u(x, y) = x^2 + y^2$.

2. (3 points) Prove that a nonconstant entire function *cannot* satisfy the two equations

   i. $f(z+1) = f(z)$
   ii. $f(z+i) = f(z)$

   for all z.

3. Find the Laurent expansion for

   (a) (1 point) $\dfrac{1}{z^4 + z^2}$ about $z = 0$

   (b) (2 points) $\dfrac{1}{z^2 - 4}$ about $z = 2$.

4. (3 points) Find the number of zeros (counting multiplicities) of $f(z) = z^4 - 5z + 1$ in $1 \leq |z| \leq 2$

5. (3 points) Evaluate the integral $\displaystyle\int_0^\infty \frac{1}{\sqrt[3]{x}(1+x)}\,dx$.

6. (3 points) Show that

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}$$

as long as $|x| < \dfrac{1}{4}$.

7. (3 points) Let R be a simply connected domain and assume $z_1, z_2 \in R$. Show that there exists a conformal mapping of R onto itself, taking z_1 to z_2. (Consider two cases: $R \neq \mathbb{C}$ and $R = \mathbb{C}$.)

8. (2 points) Evaluate the integral

$$\int_0^{2\pi} \frac{1}{1 + \tilde{\mathfrak{s}} + \tilde{\mathfrak{s}} \sin x}\, dx.$$

if ID=9521204, then

the actual problem is $\int_0^{2\pi} \frac{1}{4+3\sin x}\, dx$

$\mathfrak{s} = 9521204$

9. (2 points) Let $R \subset \mathbb{C}$ be the open set

$$R = \{z \in \mathbb{C} : |z-1| < 1 \text{ and } |z-i| < 1\}.$$

Find a conformal mapping f from R onto the unit disk U with the property

$U = \{|z|<1\}$

$$f\left(\frac{1}{\mathfrak{s}} + \frac{i}{\mathfrak{s}}\right) = 0.$$

$f = f_n \circ f_{n-1} \circ \cdots \circ f_1$

$f_j = ?$

$$f\left(\frac{1}{9521204} + i\,\frac{1}{9521204}\right) = 0$$