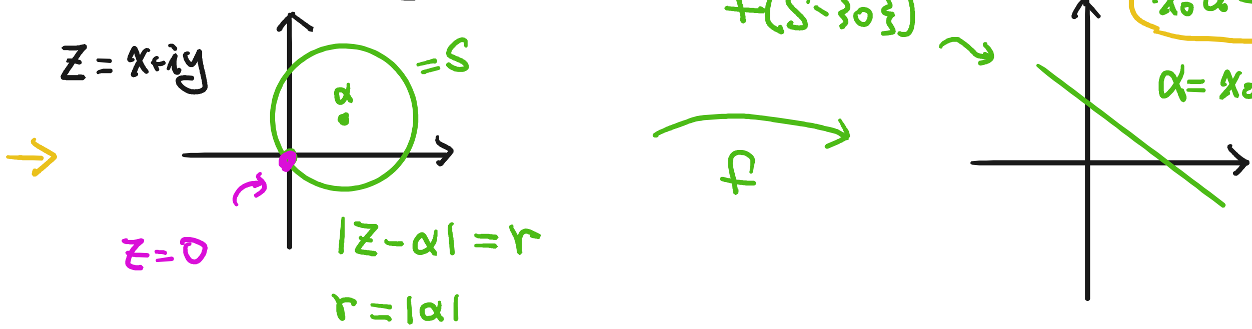
Complex Analysis 5/30

Recall

Let $f(z) = \frac{1}{z}$ $\qquad \omega = u + iv$

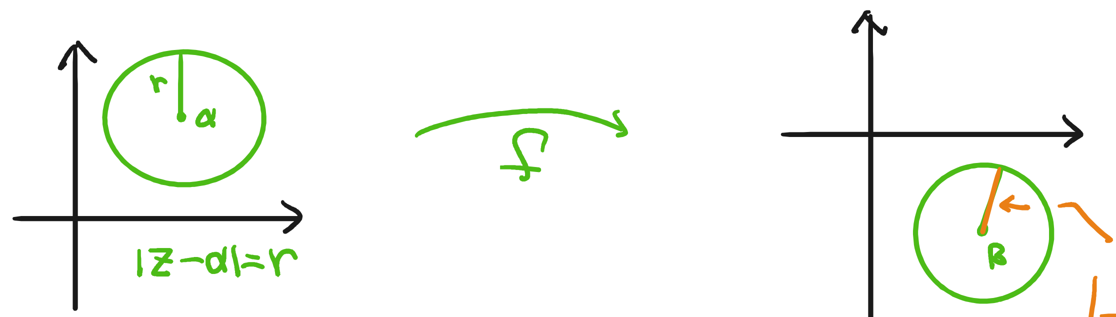


$|z - \alpha| = r$

$r = |\alpha|$

$x_0 u - y_0 v = \frac{1}{2}$

$\alpha = x_0 + iy_0$

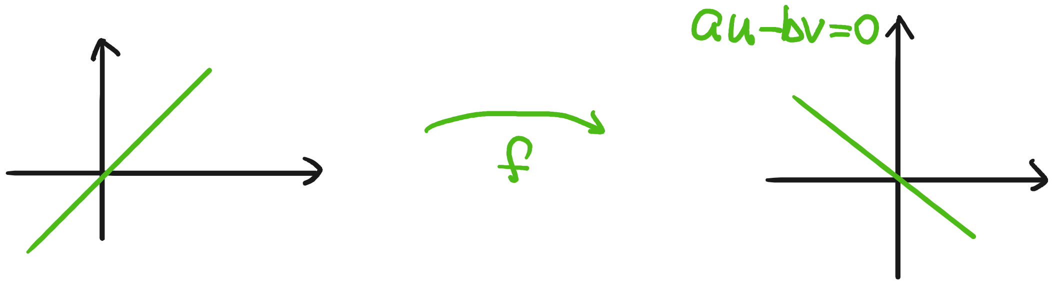


$|z - \alpha| = r$

$\left| \frac{r}{|\alpha|^2 - r^2} \right|$

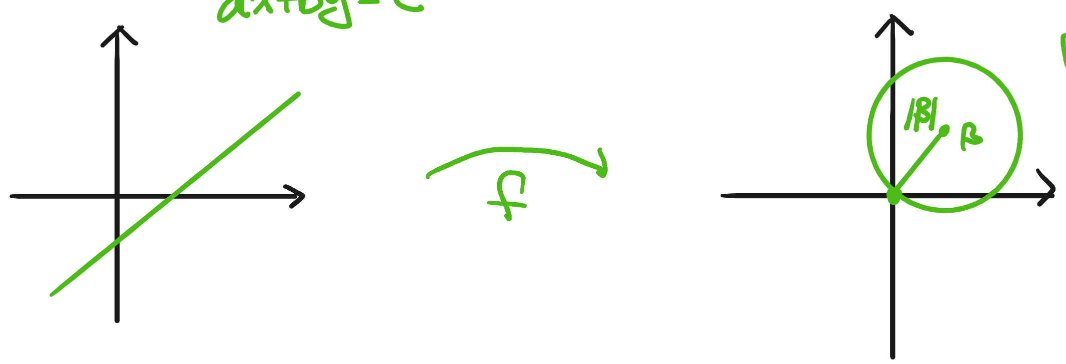
$\beta = \frac{\bar{\alpha}}{|\alpha|^2 - r^2}$

$ax + by = 0$



$au - bv = 0$

$ax + by = c$



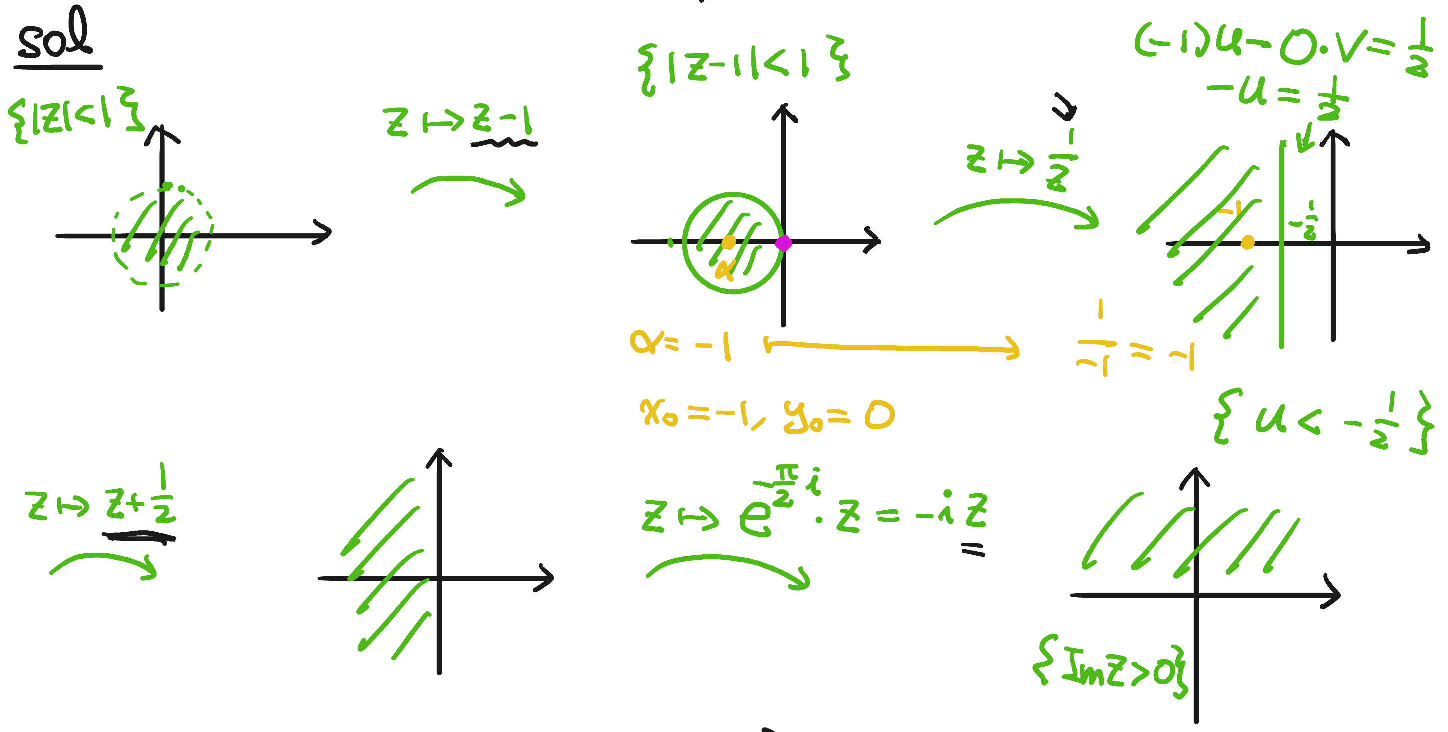
$\beta = \frac{a - bi}{2c}$

Example 1

Example 1

Find a conformal map $f : \{|z|<1\} \to \{\operatorname{Im} z > 0\}$

sol



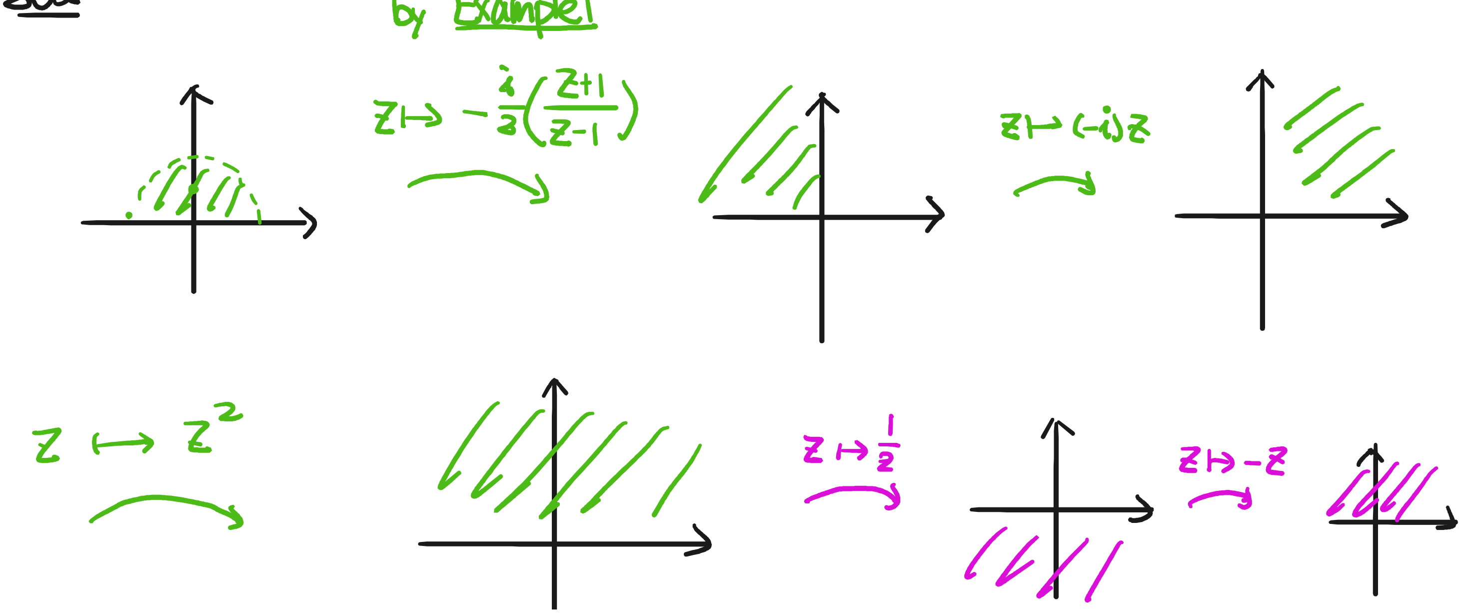
$$f(z) = -i\left(\frac{1}{z-1} + \frac{1}{2}\right) = -\frac{i}{2}\left(\frac{z+1}{z-1}\right) \quad \#$$

Note : $f(z) = -i\left(\dfrac{z+1}{z-1}\right)$ is also a correct answer!

Example2 (p.180)

Find conformal map $f : \{|z|<1,\ \operatorname{Im} z>0\} \to \{\operatorname{Im} z>0\}$

sol



$$f(z) = \left((-i)\left(-\frac{i}{2}\,\frac{z+1}{z-1}\right)\right)^2 = \frac{1}{4}\,\frac{(z+1)^2}{(z-1)^2}$$

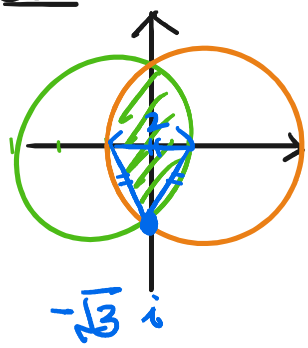
$\#$

<u>Note</u>: the answer in book is $-\frac{1}{4}\left(\frac{z-1}{z+1}\right)^2$

<u>Example 3</u>

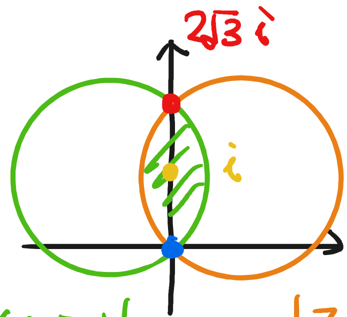
Find a conformal map $f : \{|z-1|<2,\ |z+1|<2\} \to \{|z|<1\}$

<u>Sol</u> $\{|z-1|<2,\ |z+1|<2\}$



$z \mapsto z+\sqrt{3}\,i$

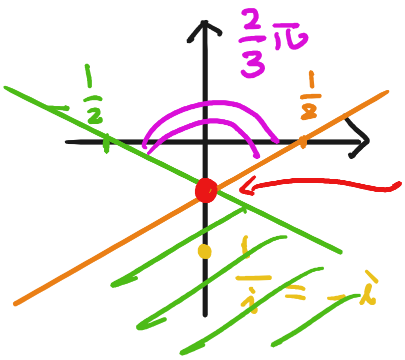
$\xrightarrow{\quad f_1 \quad}$



$|z-(-1+\sqrt{3}i)|<2$ $\qquad$ $|z-(1+\sqrt{3}i)|<2$

$z \mapsto \frac{1}{z}$

$\xrightarrow{\quad f_2 \quad}$

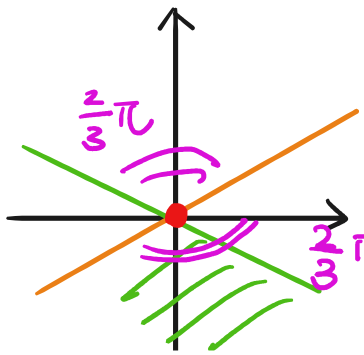


$$\frac{i}{-2\sqrt{3}} = \frac{1}{2\sqrt{3}\,i}$$

$(-1)x-\sqrt{3}y > \frac{1}{2}$ $\qquad$ $x-\sqrt{3}y > \frac{1}{2}$

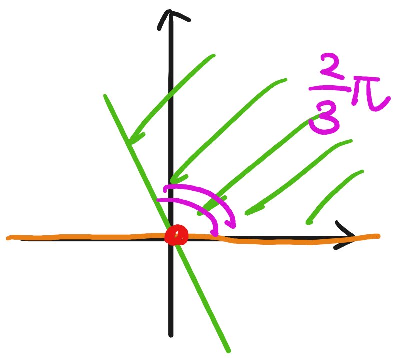
$z \mapsto z+\frac{i}{2\sqrt{3}}$

$\xrightarrow{\quad f_3 \quad}$



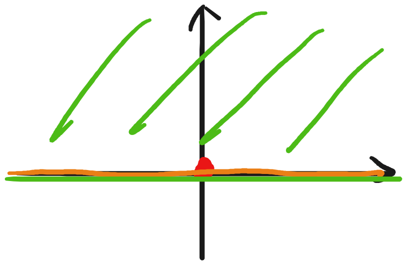
$$\frac{2}{3}\pi + \frac{\pi}{6} = \frac{5}{6}\pi$$

$$z \mapsto e^{i\frac{5}{6}\pi} z \qquad \xrightarrow{f_4}$$

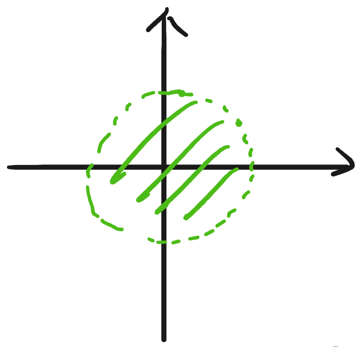


(here, $\sqrt{i} = e^{\frac{\pi}{4}i}$)

$$z \mapsto z^{\frac{3}{2}} \qquad \xrightarrow{f_5}$$



$$z \mapsto \frac{z-i}{z+i} \qquad \xrightarrow{f_6}$$



by Example 1, we can use the inverse of $-i\left(\frac{z+1}{z-1}\right) = \frac{-iz-i}{z-1}$

—— $\frac{-z+i}{-z-i} = \frac{z-i}{z+i}$

ans $= f_6 \circ f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$

#

Recall

the inverse $\frac{az+b}{cz+d}$, $ad-bc \neq 0$, is $\frac{dz-b}{-cz+a}$

($\Rightarrow B_\alpha^{-1} = B_{-\alpha}$)

Def 13.12

A conformal mapping of a region onto itself is called an automorphism of the region

Remark (13.13 – 13.14)

① If $f : D_1 \twoheadrightarrow D_2$ is conformal, then

a. any conformal mapping $D_1 \twoheadrightarrow D_2$ is of the form $g \circ f$

Notation

$\twoheadrightarrow$: onto map

$$\begin{tikzcd} D_1 \arrow[r, two heads, "f"] \arrow[dr, two heads, "\text{any Conformal } h"'] & D_2 \arrow[d, "g = h \circ f^{-1}"] \\ & D_2 \end{tikzcd}$$

$h = g \circ f$

b. any automorphism $D_1 \twoheadrightarrow D_1$ is of the form $f^{-1} \circ g \circ f$

$$\begin{tikzcd} D_1 \arrow[r, two heads, "f"] \arrow[d, "\text{any auto}"'] & D_2 \arrow[d, "g = f \circ h \circ f^{-1}"] \\ D_1 \arrow[r, two heads, "f"'] & D_2 \end{tikzcd}$$

$f^{-1} \circ g \circ f = h$

where $g : D_2 \to D_2$ is an automorphism

② The only automorphisms

$$f : \{|z|<1\} \to \{|z|<1\} \quad \text{with } f(0) = 0$$

are $f(z) = e^{i\theta} z \quad (\theta \in \mathbb{R})$

By Schwarz Lemma. See the proof of uniqueness for Riemann Mapping Thm

Thm (Thm 13.15, 13.16, 13.17)

Let $U = \{|z|<1\}$, $H^+ = \{\operatorname{Im} z > 0\}$.

① If $f : U \twoheadrightarrow U$ is an automorphism, then

$$f(z) = e^{i\theta}\left(\frac{z-\alpha}{1-\bar{\alpha}z}\right)$$

for some $|\alpha|<1$, $\theta \in \mathbb{R}$

Recall: $B_\alpha : U \to U$, $B_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$ p. 95

② If $g: H^+ \twoheadrightarrow \mathbb{U}$ is a conformal map, then

$$g(z) = e^{i\theta}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right)$$

← See Example 1 & 3.

$z \mapsto \frac{z-i}{z+i} : H^+ \to \mathbb{U}$ $(\alpha = i)$

for some $\operatorname{Im}\alpha > 0$, $\theta \in \mathbb{R}$

③ If $h: H^+ \twoheadrightarrow H^+$ is an automorphism, then

$$h(z) = \frac{az+b}{cz+d}$$

for some $a, b, c, d \in \mathbb{R}$, $ad - bc > 0$

pf (sketch, See p 183–184 for details)

① Recall: $B_\alpha : \mathbb{U} \to \mathbb{U}$, $B_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$, $|\alpha| < 1$, is an automorphism s.t. $B_\alpha(\alpha) = 0$

If $f: \mathbb{U} \to \mathbb{U}$ is an automorphism, then

$$B_{f(0)} \circ f : \mathbb{U} \xrightarrow{f} \mathbb{U} \xrightarrow{B_{f(0)}} \mathbb{U}$$

$$0 \longmapsto f(0) \longmapsto 0$$

is an automorphism s.t. $(B_{f(0)} \circ f)(0) = 0$.

By Remark ②, $B_{f(0)}(f(z)) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$

$$\Rightarrow f(z) = B_{f(0)}^{-1}(e^{i\theta} z) = B_{-f(0)}(e^{i\theta} z)$$

$$= \frac{e^{i\theta} z + f(0)}{1 + \overline{f(0)}\, e^{i\theta} z} = e^{i\theta}\left(\frac{z-\beta}{1-\bar{\beta} z}\right),$$

where $\beta = -f(0)\, e^{-i\theta}$

② <u>Step1</u> : Check $g_0(z) = \dfrac{z-i}{z+i}$ is a conformal map

$H^+ \to U$ (See Example 1 & 3)

given by formula in ①

<u>Step2</u> : By <u>Remark ①</u>, any conformal $g = \underset{U \to U}{\underline{\underline{f}}} \circ g_0$

and compute #

③
$$\begin{tikzcd} H^+ \arrow[r, "h", "\text{any}"'] \arrow[d, "g_0"'] & H^+ \arrow[d, "g_0"] \\ U \arrow[r, "f \text{ by } ①"'] & U \end{tikzcd}$$
$\Rightarrow h = g_0^{-1} \circ f \circ g_0$

(Remark ①) #

<u>Thm 13.23</u>

The unique bilinear transformation $\omega = f(z)$ mapping z_1, z_2, z_3 to $\omega_1, \omega_2, \omega_3$, respectively, is given by

$$\frac{(\omega-\omega_2)(\omega_3-\omega_1)}{(\omega-\omega_1)(\omega_3-\omega_2)} = \frac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)}$$

pf: skip.

<u>Example</u>

Find a bilinear transformation mapping

$0 \mapsto i$, $-1 \mapsto 0$, $i \mapsto -1$

<u>sol</u>

$$\frac{(\omega-0)(-1-i)}{(\omega-i)(-1-0)} = \frac{(z-(-1))(i-0)}{(z-0)(i-(-1))}$$

$$\begin{array}{ccc} \| & \qquad\qquad & \| \\ \dfrac{\omega(1+i)}{\omega - i} & & \dfrac{i}{i+1}\,\dfrac{z+1}{z} \end{array}$$

$$\Rightarrow \quad \omega z(1+i)^2 = i(\omega - i)(z+1)$$

$$\omega = \frac{z+1}{iz - i} = -i\,\frac{z+1}{z-1} \qquad \#$$