Complex Analysis 5/26

Recall (Riemann Mapping Thm)

For any simply connected domain R which is not $\mathbb{C}$ and $z_0 \in R$, there exists a unique conformal map $\varphi : R \xrightarrow[\text{onto}]{1-1} \underbrace{U}_{\{|z|<1\}}$ unit disc

s.t.

- $\varphi(z_0) = 0$,
- $\varphi'(z_0)$ is a positive real number

Remark

- Proof of uniqueness is easy: by Schwarz Lemma (Lemma 7.2)
- Proof of existence is difficult: Skip. See p. 201 - p. 204.

  Instead, we will show explicit conformal maps between various simply connected domains.

exer

Use Riemann Mapping Thm to show that $\mathbb{C}$ is NOT conformally equivalent to any simply connected domain $R \neq \mathbb{C}$.

Note: $U = \{|z|<1\}$

$\Rightarrow$ entire

If $\varphi : \mathbb{C} \longrightarrow U$ is a conformal map, then

$|\varphi(z)| < 1 \quad \forall z \in \mathbb{C}$

bounded

$\Rightarrow$ φ is constant $(\longrightarrow\longleftarrow)$

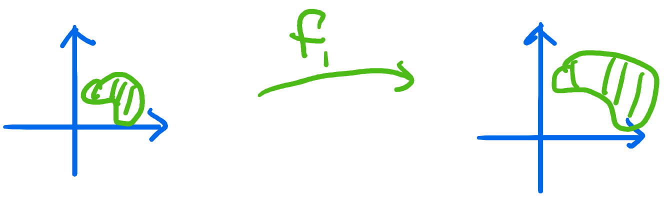
Explicit conformal maps

I. Elementary transformations:

(i) $\omega = az+b$, $a \neq 0$, is the composition of 3 maps

1. $f_1(z) = kz$, $k = |a| > 0$

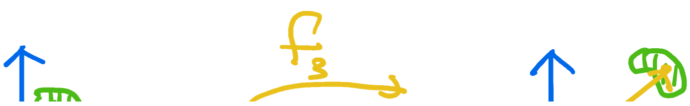
— magnification



2. $f_2(z) = e^{i\theta} z$, $\theta = \operatorname{Arg} a$

— rotation

3. $f_3(z) = z+b$ — translation



That is, $\omega = az+b = f_3(f_2(f_1(z)))$

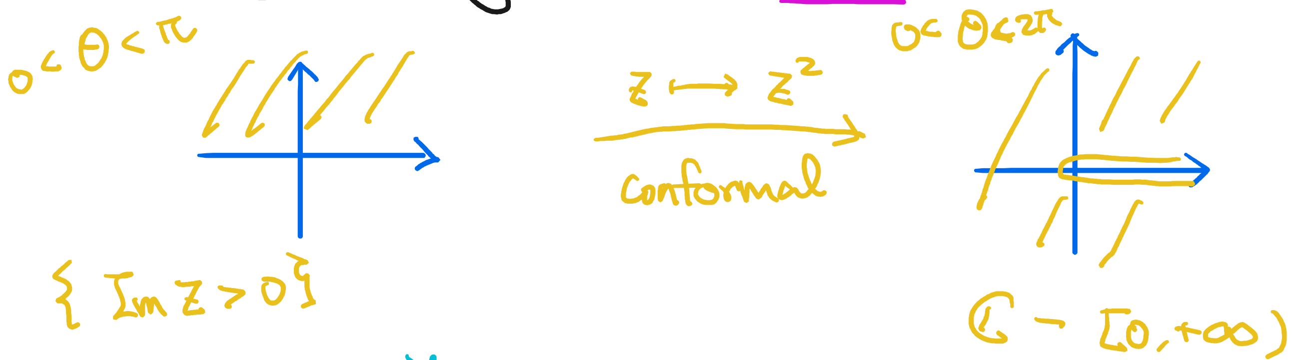
(ii) $\omega = z^{\alpha}$, $\alpha > 0$, given by $e^{\alpha \log z}$

is analytic in a simply connected domain $\not\ni 0$.

If $z = re^{i\theta}$, then $\omega = r^{\alpha} e^{i\alpha\theta}$

which is conformal in $\sqrt{z}$ is NOT differentiable at $z=0$

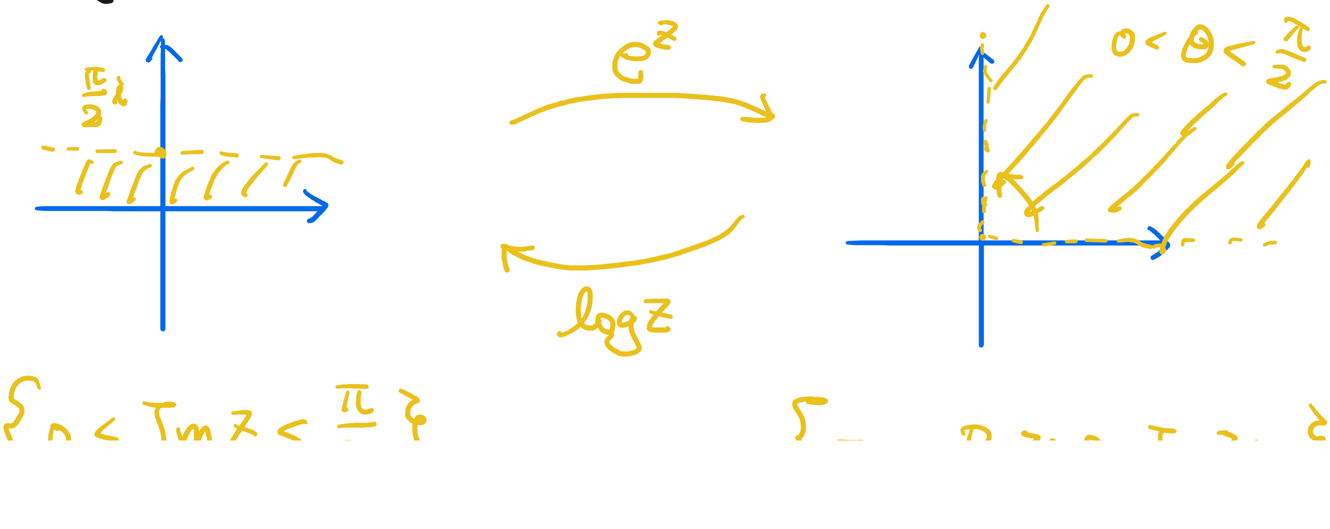
$\{ \theta_1 < \operatorname{Arg} z < \theta_2 \ , \ z \neq 0 \}$. $\theta_2 - \theta_1 \leqslant \frac{2\pi}{\alpha}$



(iii) $\omega = e^{z} = e^{x} e^{iy} = e^{x}(\cos y + i \sin y)$, $z = x+iy$

is conformal in

$\{ y_1 < y < y_2 \}$, $y_2 - y_1 \leqslant 2\pi$



$\{z : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$
$= \{z = re^{i\theta} : r \in (0, +\infty), 0 < \theta < \frac{\pi}{2}\}$

II. Bilinear transformations (a.k.a. Möbius transformations)

The map given by

$$f(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0$$

is called a bilinear transformation / Möbius transformation

Recall (group action)
$G \times S \to S$
$\{\text{Möbius}\} \times S^1 = \mathbb{C} \cup \{\infty\} \to S^1$

Remark

① $\omega = \frac{az+b}{cz+d} = f(z) \Rightarrow \omega(cz+d) = az+b$

$$\Rightarrow \quad z = \frac{d\omega - b}{-c\omega + a} \qquad \forall \ \omega \neq \frac{a}{c}$$

So $f : \mathbb{C} \setminus \{-\frac{d}{c}\} \xrightarrow[\text{onto}]{1-1} \mathbb{C} \setminus \{\frac{a}{c}\}$

is conformal

② $B_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$ (p. 95) is an example of bilinear transformation

$a = 1, \ b = -\alpha$
$c = -\bar{\alpha}, \ d = 1$
$ad - bc = 1 - \alpha \cdot \bar{\alpha} = 1 - |\alpha|^2 > 0$ if $|\alpha| < 1$

$ad - bc = 1 - \dots$

③ If $g(z) = \dfrac{\tilde{a} z + \tilde{b}}{\tilde{c} z + \tilde{d}}$, $\tilde{a}\tilde{d} - \tilde{b}\tilde{c} \neq 0$, then

$$(f \circ g)(z) = f\left(\frac{\tilde{a}z+\tilde{b}}{\tilde{c}z+\tilde{d}}\right) = \frac{a \cdot \frac{\tilde{a}z+\tilde{b}}{\tilde{c}z+\tilde{d}} + b}{c \cdot \frac{\tilde{a}z+\tilde{b}}{\tilde{c}z+\tilde{d}} + d}$$

$$= \frac{(a\tilde{a}+b\tilde{c})z + (a\tilde{b}+b\tilde{d})}{(c\tilde{a}+d\tilde{c})z + (c\tilde{b}+d\tilde{d})}$$

bilinear transformation

$$\leftrightarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = \begin{pmatrix} a\tilde{a}+b\tilde{c} & a\tilde{b}+b\tilde{d} \\ c\tilde{a}+d\tilde{c} & c\tilde{b}+d\tilde{d} \end{pmatrix}$$

$ad - bc \neq 0$
$= \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\det \neq 0 \Rightarrow$

$\det = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \det\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \neq 0$

So $\left\{ \dfrac{az+b}{cz+d} : ad - bc \neq 0 \right\}$ $\left(\cong GL_2(\mathbb{C}) / \mathbb{C}^* \cdot I \right)$

is a group under composition. $\dfrac{r \cdot az + rb}{r \cdot cz + r \cdot d} = \dfrac{az+b}{cz+d}$

<u>Lemma 13.10</u>

If S is a circle or a line, and $f(z) = \frac{1}{z}$,

then $f(S)$ is also a circle or a line.

pf (see exer 27, 28 in Ch 1 for a different proof)

① Let

$$S = C(\alpha; r) = \{\alpha + re^{i\theta} : \theta \in [0, 2\pi]\}$$

$$f(S) = \{ \omega = \frac{1}{z} : z \in S,\ z \neq 0 \}$$

Note:

$$C(\alpha; r) = \{ |z-\alpha|^2 = r^2 = (z-\alpha)(\bar{z}-\bar{\alpha}) \}$$

$$\Leftrightarrow\ z\bar{z} - \alpha\bar{z} - \bar{\alpha}z = r^2 - |\alpha|^2$$

$(\omega = \frac{1}{z})$

or

$$(*)\quad \frac{1}{\omega\bar{\omega}} - \frac{\alpha}{\bar{\omega}} - \frac{\bar{\alpha}}{\omega} = r^2 - |\alpha|^2$$

<u>case 1</u> $r = |\alpha|$, i.e. $0 \in S = C(\alpha; r)$

$$(*):\quad 1 - \alpha\omega - \bar{\alpha}\bar{\omega} = 0$$

$$= 1 - 2\,\mathrm{Re}(\alpha\omega)$$

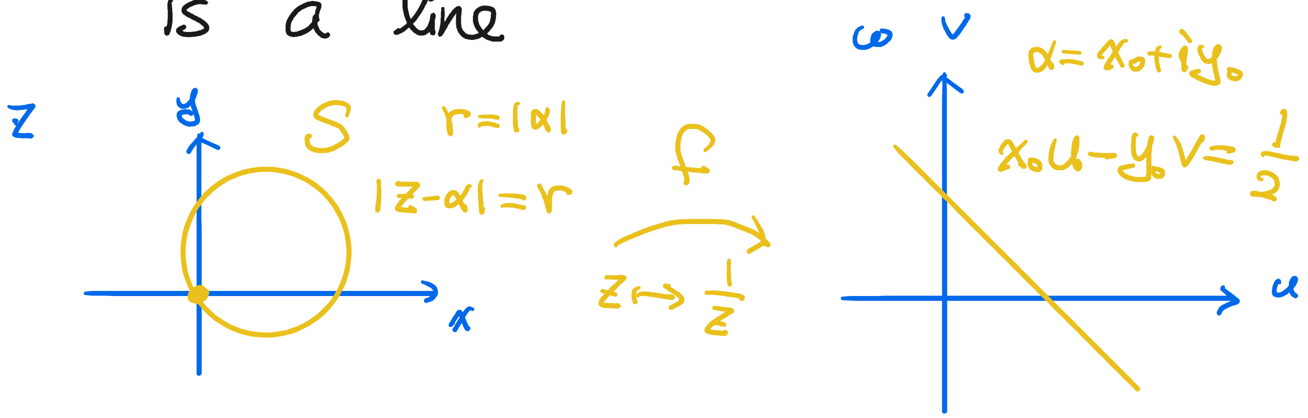
$$\Leftrightarrow\quad \mathrm{Re}(\alpha\omega) = \frac{1}{2}$$

If $\alpha = x_0 + iy_0$, $\omega = u + iv$, then

$$\mathrm{Re}(\alpha\omega) = x_0 u - y_0 v = \frac{1}{2}$$

$$\Rightarrow\quad f(S) = \{ \omega = u + iv \in \mathbb{C} : x_0 u - y_0 v = \frac{1}{2} \}$$

is a line



Case 2: $r \neq |\alpha|$, i.e. $0 \notin S$

$(*)$: $$\omega\bar{\omega} - \left(\frac{\bar{\alpha}}{|\alpha|^2 - r^2}\right)\bar{\omega} - \left(\frac{\alpha}{|\alpha|^2 - r^2}\right)\omega = \frac{-1}{|\alpha|^2 - r^2}$$

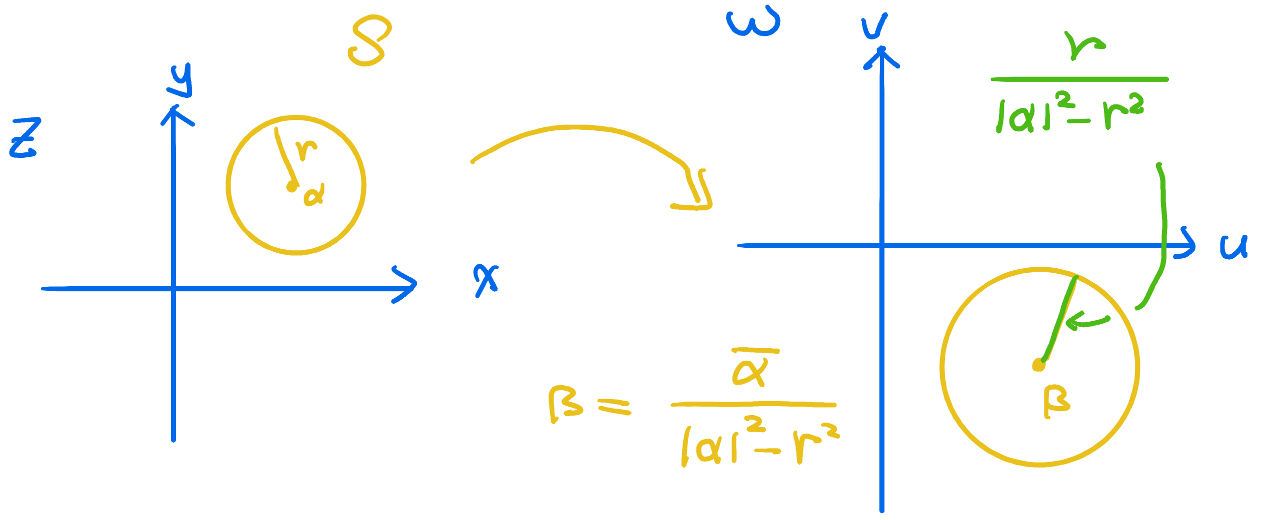
Set

$$\beta = \frac{\bar{\alpha}}{|\alpha|^2 - r^2}$$

$$\frac{|\alpha|^2}{(|\alpha|^2 - r^2)^2} \qquad \frac{-|\alpha|^2 + r^2}{(|\alpha|^2 - r^2)^2}$$

$(*) \Leftrightarrow$ $$\omega\bar{\omega} - \beta\bar{\omega} - \bar{\beta}\omega + |\beta|^2 = \frac{r^2}{(|\alpha|^2 - r^2)^2}$$

$$= |\omega - \beta|^2 \qquad \left(\frac{r}{|\alpha|^2 - r^2}\right)^2$$



② If S is a line, then $\exists\, a, b, c \in \mathbb{R}$ s.t.

$(**)$ $\quad z = x + iy \in S \Rightarrow \underline{ax + by = c}$

Let

$\alpha = a - bi \Rightarrow \underline{\mathrm{Re}(\alpha z)} = ax + by$

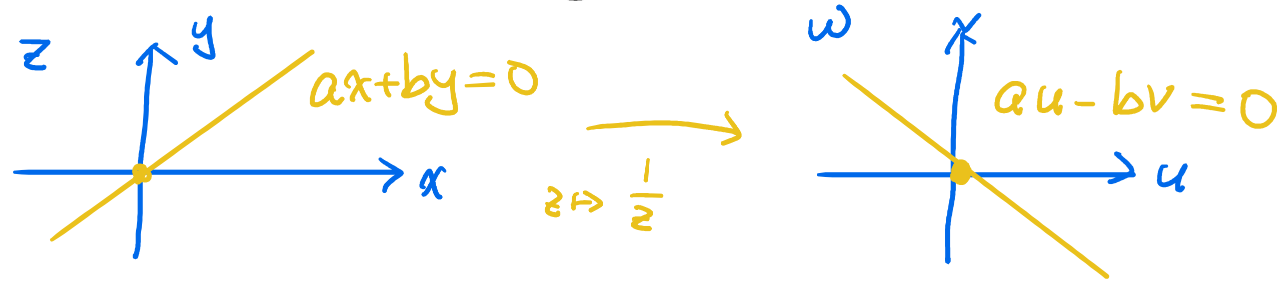
$(**) \Leftrightarrow \quad \mathrm{Re}(\alpha z) = c \quad$ or $\quad \alpha z + \bar{\alpha}\bar{z} = 2c$

<u>case 1</u>: $c=0$, i.e. $0 \in S$

If $\omega = u+iv$, then

$$\alpha z = \frac{\alpha}{\omega} = \frac{\alpha(u-iv)}{u^2+v^2}$$

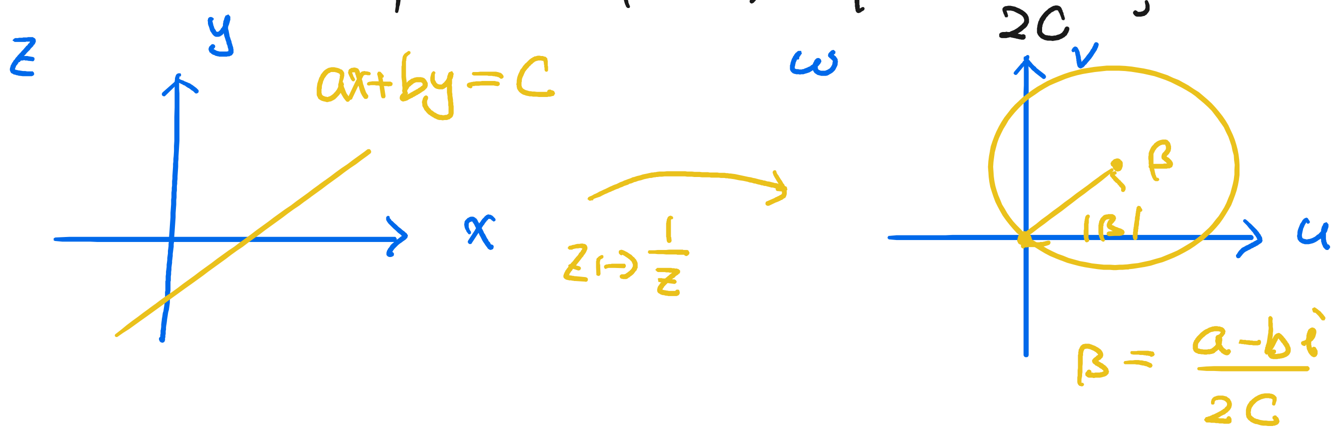
(**): $au - bv = 0$ —— line



<u>case 2</u>: $c \neq 0$, i.e. $0 \notin S$.

(**): $$\omega\bar{\omega} - \frac{\alpha}{2c}\bar{\omega} - \frac{\bar{\alpha}}{2c}\omega = 0$$

$$\Rightarrow |\omega - \beta|^2 = |\beta|^2, \quad \beta = \frac{\alpha}{2c}, \text{ —— circle}$$



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<u>Thm 13.11</u>

$$f(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0,$$

maps circles and lines onto circles and lines circles and lines

pf

① If $c=0$, then f is linear and the result is immediate (see elementary transformations)

② If $c \neq 0$,

$$f(z) = \frac{az+b}{cz+d} = \frac{1}{c}\left(a - \frac{ad-bc}{cz+d}\right)$$

$$= (f_3 \circ f_2 \circ f_1)(z)$$

where

Lemma 13.10

linear

$\to$ $f_1(z) = cz+d$, $f_2(z) = \frac{1}{z}$,

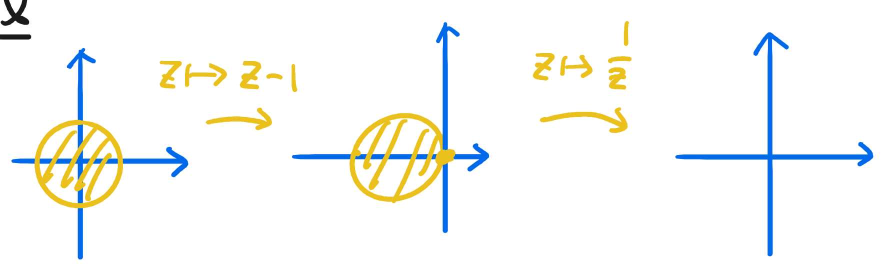
$\to$ $f_3(z) = \frac{a}{c} - \frac{ad-bc}{c} z$

$\Rightarrow$ the result follows. #

Example

Find a conformal map $f: \{|z|<1\} \to \{\operatorname{Im} z > 0\}$

sol



... explain next Monday.