Complex Analysis 5/23

Recall (local behavior of analytic function)

Let f be analytic at $z_0 \in \mathbb{C}$.
Let k be the least positive integer for which $f^{(k)}(z_0) \neq 0$

(if $\nexists$ such k, i.e. $f^{(l)}(z_0) = 0 \ \forall l \in \mathbb{N}$, then f is constant near z_0)

- (Thm 13.4) If $k = 1$, i.e. $f'(z_0) \neq 0$, then f is locally 1-1 and conformal at z_0.

  保角

- (Thm 13.7) If $k > 1$, then f is k-to-1 and f magnifies angles at z_0 by k in a nbd of z_0

  correction

i.e. $f(z_0) = \alpha = f(z_1)$
i.e. z_0, z_1 are roots of $f(z) = \alpha$

e.g. $f(z) = z^k$

$|z| < \delta \Rightarrow |z^k| < \delta^k$

$D(0;\delta) \to D(0;\delta^k)$

$z^k = \alpha$ has k roots

pf of Thm 13.7

① Without loss of generality, assume $f(z_0) = 0$

(if not, replace $f(z)$ by $f(z) - f(z_0)$)

By hypothesis, the Taylor expansion of f about z_0 is the form

$$f(z) = (z-z_0)^k \underbrace{\left(a_k + a_{k+1}(z-z_0) + a_{k+2}(z-z_0)^2 + \cdots\right)}_{\overset{!!}{g(z)}}$$

where $a_k = \dfrac{f^{(k)}(z_0)}{k!} \neq 0$

② Since $g(z_0) = a_k \neq 0$, $\exists\, \delta > 0$ s.t.

$$g(z) \neq 0 \quad \forall z \in D(z_0;\delta) \leftarrow \text{simply connected}$$

Recall $\log g(z)$ can be defined analytically in a simply connected domain where $g(z) \neq 0$

$\int_{z_0}^{z} \frac{g'(z)}{g(z)}\,dz$

$$\Rightarrow \exists\, g^{1/k}(z) = e^{\frac{1}{k}\log g(z)} \text{ analytic in } D(z_0;\delta)$$

s.t.

$$\left(g^{1/k}(z)\right)^k = g(z) \quad \forall z \in D(z_0;\delta)$$

③ Let

$$h(z) = (z-z_0)\, g^{1/k}(z) \quad \text{— analytic in } D(z_0;\delta)$$

$$\Rightarrow f(z) = (h(z))^k = (z-z_0)^k g(z)$$

$$h(z_0) = 0$$

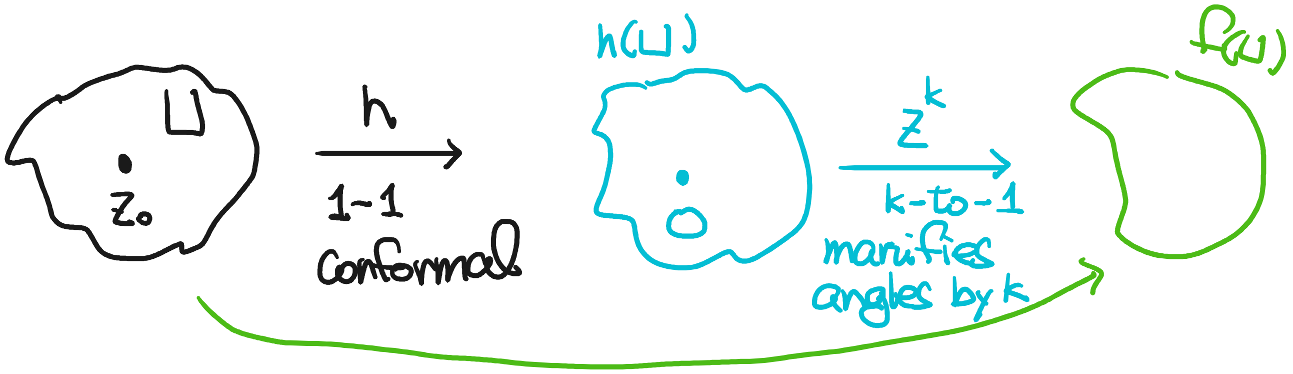
$$h'(z_0) = g^{1/k}(z_0) + (z_0-z_0)(g^{1/k})'(z_0)$$

$$= g^{1/k}(z_0) \neq 0$$

Thm 13.4

$\Rightarrow \exists$ nbd U of z_0 in $D(z_0;\delta)$ s.t.

h is 1-1, conformal in U



$(h(z))^k = f(z)$

k-to-1
manifies angles by k $\#$

Thm 13.8

Suppose f is a 1-1 analytic function in a region D. Then

a. f^{-1} exists and is analytic in $f(D)$ — open by Open Mapping Thm

b. f and f^{-1} are conformal in D and $f(D)$, respectively

pf: last time

Def 13.9

a. A 1-1 analytic mapping is called a conformal mapping

b. Two regions D_1 and D_2 are conformally equivalent if $\exists$ conformal mapping

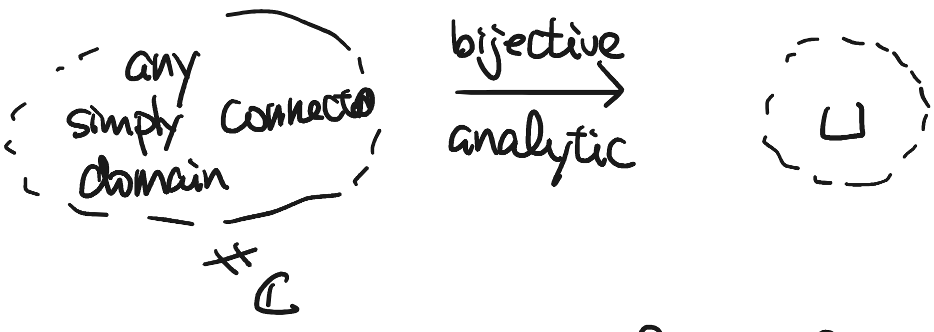
$D_1 \xrightarrow[\text{onto}]{1-1} D_2$.

Such a bijective conformal mapping is called a conformal equivalence or biholomorphism.

Riemann Mapping Theorem (see §14.2)

Any two simply connected domains (except $\mathbb{C}$) are conformally equivalent.

Note that if we have a conformal mapping from any simply connected domain $\neq \mathbb{C}$ onto unit disc $U = \{|z|<1\}$



then we can get a conformal equivalence between any two simply connected domains (except $\mathbb{C}$) R_1, R_2 by composition

$$\begin{tikzcd} R_1 \arrow[r, "f_1"] \arrow[rr, bend right, "f_2^{-1}\circ f_1"'] & \mathbb{U} & R_2 \arrow[l, "f_2"'] \end{tikzcd}$$

In fact, one has the following (better) formulation of Riemann Mapping Thm:

Riemann Mapping Theorem (p.200)

For any simply connected domain R which is not $\mathbb{C}$ and any $z_0 \in R$, there exists a unique conformal equivalence $\varphi : R \longrightarrow \mathbb{U} = \{|z|<1\}$ s.t.

$$\varphi(z_0) = 0 \qquad \text{and} \qquad \underbrace{\varphi'(z_0) > 0}$$

it means $\varphi'(z_0) \in \mathbb{R} \subsetneq \mathbb{C}$ and $\varphi'(z_0) > 0$

Remark

① For $\alpha \in \mathbb{U}$, we considered the function

$$B_\alpha : \mathbb{U} \longrightarrow \mathbb{U}, \quad B_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha} z}$$

which is (i) analytic, (ii) $B_\alpha(\alpha) = 0$

p.95 (iii) $B_\alpha'(\alpha) = \dfrac{(1-\bar{\alpha} z) - (z-\alpha)(-\bar{\alpha})}{} \Big|$

p. ... Ch7

$$(1-\bar{\alpha} z)^2 \Big|_{z=\alpha}$$

$$= \frac{1-|\alpha|^2}{(1-|\alpha|^2)^2} = \frac{1}{1-|\alpha|^2} > 0$$

(iv) Note $B_\alpha(z) = \omega = \dfrac{z-\alpha}{1-\bar{\alpha} z} \Rightarrow z-\alpha = \omega - \bar{\alpha} z \omega$

See p.95, Ch7 $\Rightarrow z = \dfrac{\omega+\alpha}{1+\bar{\alpha}\omega} = B_{-\alpha}(\omega)$

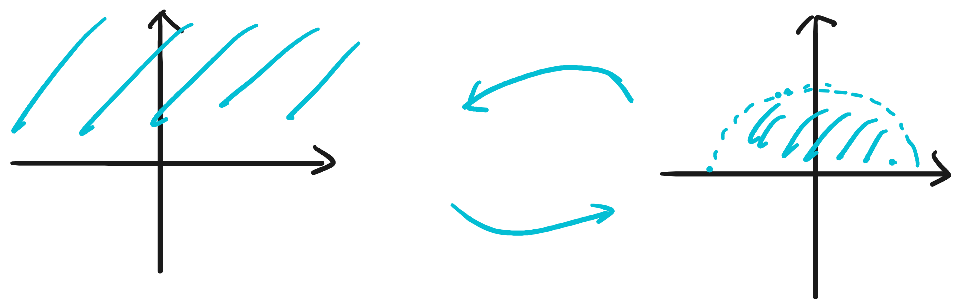
So $B_\alpha : U \to U$ has the inverse function $B_{-\alpha}$ ($\Rightarrow B_\alpha : U \to U$ is 1-1, onto)

Conclusion:

$B_\alpha : U \to U$ is the unique conformal equivalence s.t. $B_\alpha(\alpha) = 0$, $B_\alpha'(\alpha) > 0$.

② Problem 10 in HW6:

Use the conformal map



to transfer Schwartz reflection principle (Thm 7.8)

③ The proof of uniqueness is easy:

Recall (Schwarz Lemma, Lemma 7.2)

Suppose $f: \mathbb{U} \to \mathbb{U}$ is analytic, $f(0)=0$. Then

(i) $|f(z)| \le z \quad \forall z \in \mathbb{U}$

(ii) $|f'(0)| \le 1$

And "=" holds in either (i) or (ii) $\Leftrightarrow f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$

Suppose $\varphi_1, \varphi_2 : R \longrightarrow \mathbb{U}$ satisfy the properties in Thm.

$$\Rightarrow \Phi := \varphi_1 \circ \varphi_2^{-1} : \mathbb{U} \longrightarrow \mathbb{U} \text{ is analytic and}$$

$0 \to z_0 \to 0$

$\Phi^{-1} = \varphi_2 \circ \varphi_1^{-1}$

$$\Phi(0) = 0$$

By Schwarz Lemma, $|\Phi(z)| \le |z|$

Similarly, $|\Phi^{-1}(z)| \le |z|$

$$\Rightarrow |\Phi(z)| \le |z| = |\Phi^{-1}(\Phi(z))| \le |\Phi(z)|$$

$$\Rightarrow |\Phi(z)| = |z|$$

Schwarz $\Rightarrow$ Lemma $\Phi(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$

But $\Phi'(0) = e^{i\theta} > 0 \quad (\because \varphi'(z_0), \varphi'(z_0) > 0)$

$$\Rightarrow \quad e^{i\theta} = 1 \quad \Rightarrow \quad \Phi(z) = z$$

$$\Rightarrow \quad \varphi_1 = \varphi_2$$ #