Complex Analysis 5/19

Summary of Ch 10-12

- Computation of residue
  - if $f = \frac{A}{B}$, B has a simple zero at z_0 and $A(z_0) \neq 0$, then

$$\mathrm{Res}(f; z_0) = \frac{A(z_0)}{B'(z_0)}$$

e.g. $\mathrm{Res}\left(\frac{z}{z-1}; 1\right) = \frac{z}{(z-1)'}\Big|_{z=1} = 1$

  - if f is not the above case, then find the Laurent expansion for f about z_0

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$

$$\Rightarrow \mathrm{Res}(f; z_0) = c_{-1}$$

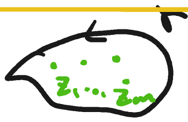
e.g. Problem 6, Exam 3

- Residue Thm (see Thm 10.5 for details)

γ: closed curve not passing singularities

easy way to compute integral →

$$\boxed{\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{m} \underbrace{n(\gamma, z_k)}_{\text{winding number}} \mathrm{Res}(f; z_k)}$$

where 

- Computation of integration, sum (Ch 11-12)

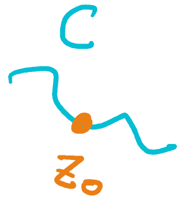
<u>idea</u>: find a good γ (closed curve)

and (i) show $\int_\gamma f(z)dz \longrightarrow$ answer

(ii) Compute $\int_\gamma f(z)dz$ by Residue Thm.

Ch 13-14 Conformal mapping 保角

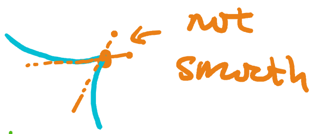
conformal equivalence



Def 13.1 regard C as a subset in $\mathbb{C}$

Let C be a curve in $\mathbb{C}$ and $z_0 \in C$.

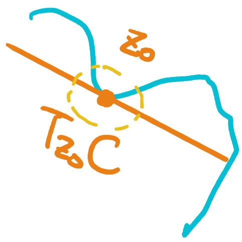
We say C is smooth at z_0 if $\exists$ smooth parametrization $z : (-\varepsilon, \varepsilon) \longrightarrow C$ $c^{\mathbb{C}}$ s.t.



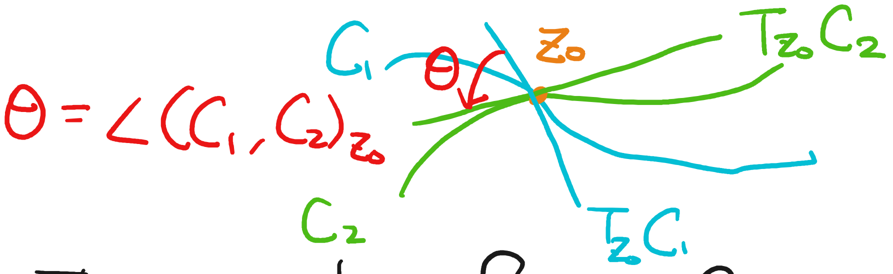
$z(0) = z_0$, $z'(0) \neq 0$. ← "slope" of tangent line

In this case, the tangent line of C at z_0 is the line

$$T_{z_0}C := \{ z_0 + t\, z'(0) : t \in \mathbb{R} \}$$



Let C_1 and C_2 be 2 curves which are smooth and intersects at z_0



The angle from C_1 to C_2 at z_0,

denoted $\angle(C_1, C_2)_{z_0}$, is defined as the angle measured counterclockwise from $T_{z_0}C_1$ to $T_{z_0}C_2$

Def 13.2

Suppose f is defined in a nbd of z_0. f is said to be conformal at z_0 if "f preserves angles at z_0"

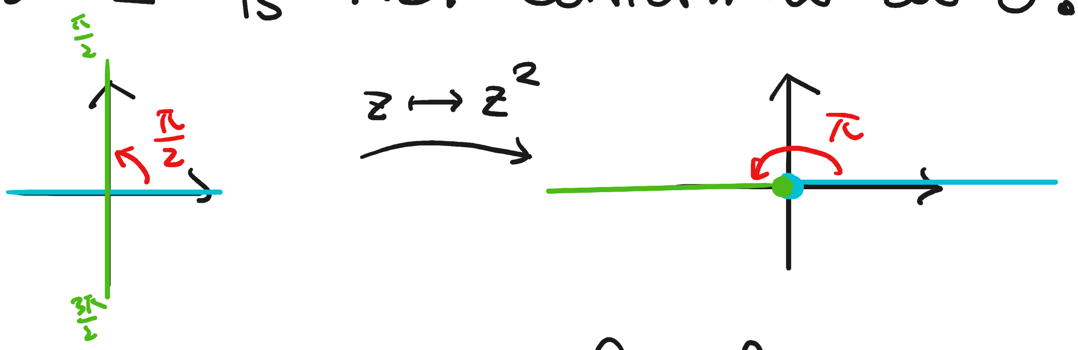
That is, for each pair of curves C_1, C_2 smooth at z_0, intersecting at z_0, one has

$$\angle(C_1, C_2)_{z_0} = \angle(f(C_1), f(C_2))_{f(z_0)}$$

We say f is conformal in a region D if f is conformal at all points $z \in D$.

Example

- $f(z) = z$ is conformal in $\mathbb{C}$
- $g(z) = z^2$ is NOT conformal at 0:



(But, in fact, g is conformal in $\mathbb{C} \setminus \{0\}$)

<u>Def 13.3</u>

a. f is 1-1 function in a region D if for every $z_1 \neq z_2$ in D, $f(z_1) \neq f(z_2)$

b. f is locally 1-1 at z_0 if f is 1-1 in a nbd of z_0

c. f is locally 1-1 throughout a region D if f is locally 1-1 at every point $z \in D$.

<u>exer</u>

$g(z) = z^2$ is (i) NOT locally 1-1 at 0
(ii) NOT 1-1 in $\mathbb{C} \setminus \{0\}$
(iii) locally 1-1 in $\mathbb{C} \setminus \{0\}$

<u>Thm 13.4</u> (cf. Inverse Function Thm)

Suppose f is analytic at z_0 and $f'(z_0) \neq 0$. Then f is conformal and locally 1-1 at z_0.

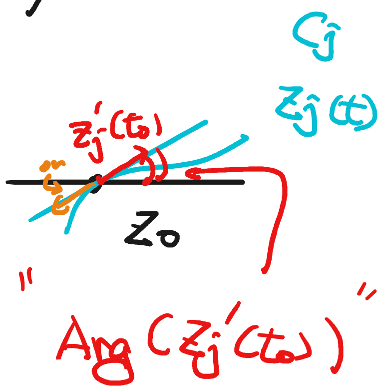
pf

① Let C_j be parametrized by ($j = 1, 2$)

$$z_j(t) = x_j(t) + i y(t), \quad z_j(t_0) = z_0$$

$\Rightarrow$ $f(C_j)$ is parametrized by

$$w_j(t) = f(z_j(t))$$



$$\Rightarrow \mathrm{Arg}\, w_j'(t_0)$$

chain rule

$$= \mathrm{Arg}(f'(z_0) \cdot z_j'(t_0))$$

$$= \mathrm{Arg}(f'(z_0)) + \mathrm{Arg}(z_j'(t_0))$$

$$\Rightarrow \angle(C_1, C_2)_{z_0} = \mathrm{Arg}(z_2'(t_0)) - \mathrm{Arg}(z_1'(t_0))$$

$$= (\mathrm{Arg}\, f'(z_0) + \mathrm{Arg}\, z_2'(t_0)) - (\mathrm{Arg}\, f'(z_0) + \mathrm{Arg}\, z_1'(t_0))$$

$$= \mathrm{Arg}\, w_2'(t_0) - \mathrm{Arg}\, w_1'(t_0)$$

$$= \angle(f(C_1), f(C_2))_{f(z_0)}$$

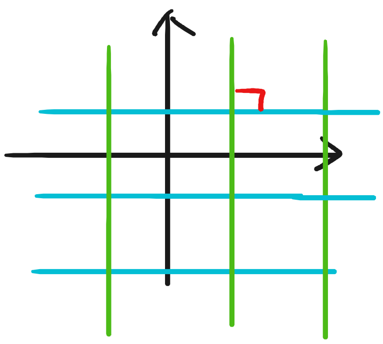
$\Rightarrow f$ is conformal at z_0.

② (Recall the total differential
$$Df(z_0)(z-z_0) = \underbrace{f'(z_0)}_{\neq 0 \ \Rightarrow\ Df(z_0) \text{ is invertible}} \cdot (z-z_0))$$

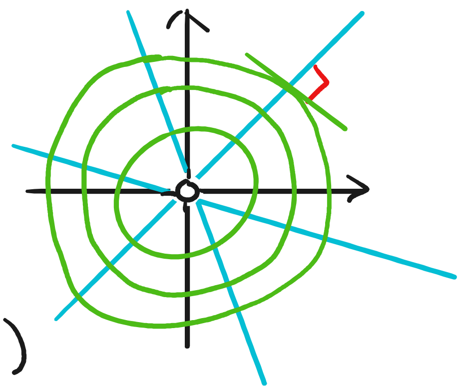
By Inverse Function Function, f is locally 1-1 at z_0 #

(See p. 170 – 171 for a different proof for locally 1-1)

Example 1 $\quad ((e^z)' = e^z \neq 0)$



$\xrightarrow{e^z}$
Conformal
locally 1-1
(but not 1-1)



Example 2

Let $g(z) = z^2 \Rightarrow g'(z) = 2z \neq 0 \quad \forall z \neq 0$

$\Rightarrow g$ is conformal, locally 1-1 throughout $\mathbb{C}\setminus\{0\}$
(NOT 1-1)

see p. 172 for picture

Def 13.5

Let $k \in \mathbb{N}$. We say f is a k-to-1 mapping

Let $k \in \mathbb{N}$. We say ... a ... mapping of D_1 onto D_2 if $\forall\, \alpha \in D_2$, the equation $f(z) = \alpha$ has k roots (counting multiplicities) in D_1.

Example

Let $f(z) = z^k$, $k \in \mathbb{N}$, $\delta > 0$. Then

$$f : D(0;\delta) \longrightarrow D(0;\delta^k)$$

is k-to-1.

Thm 13.7 (cf. Thm 13.4)

Suppose f is analytic at z_0 with $f'(z_0) = 0$. If f is not constant in a nbd of z_0, then f is a k-to-1 mapping in a nbd of z_0 and f magnifies angles at z_0 by k, where k is the least positive integer for which $f^{(k)}(z_0) \neq 0$ $(k \neq 1)$

更正

pf: next time

Thm 13.8

Suppose f is a 1-1 analytic function in a region D. Then

a) f^{-1} exists and is analytic in $f(D)$

a. f^{-1} exists and is analytic in $f(D)$

b. f and f^{-1} are conformal in D and $f(D)$, respectively

pf

f is 1-1 $\overset{\text{Thm 13.7}}{\Longrightarrow}$ $f' \neq 0$

$\overset{\text{Prop 3.5}}{\Longrightarrow}$ f^{-1} is also analytic and

$(f^{-1})' = \frac{1}{f'} \neq 0$

$\overset{\text{Thm 13.4}}{\Longrightarrow}$ both f and f^{-1} are conformal.

#

$$\frac{1}{z^4+z^2} = \frac{1}{z^2}\frac{1}{z^2+1} = \frac{1}{z^2}\cdot\frac{1}{1-(-z^2)}$$

$$= \frac{1}{z^2}\cdot\sum_{n=0}^{\infty}(-z^2)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n z^{2n-2}$$

$2n-2 \neq -1 \quad \forall n \in \mathbb{Z}$

$\Rightarrow C_{-1} = 0$

6. Find the Laurent expansion for

$\mathrm{Res}\left(\frac{1}{z^4+z^2}; 0\right) = ?$ 0

(a) (2 points) $\dfrac{1}{z^4+z^2}$ about $z=0$

(b) (3 points) $\dfrac{1}{z^2-4}$ about $z=2$. $\displaystyle\sum_{n=-\infty}^{\infty} C_n (z-2)^n$

(b)

$$\frac{1}{(z-2)(z+2)} = \frac{1}{z-2}\,\frac{1}{(z-2)+4}$$

$$= \frac{\frac{1}{4}}{1-\frac{2-z}{4}}$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{2-z}{4}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}(z-2)^{n-1}$$

$$\frac{1}{z^2-4} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{4^{n+2}}(z-2)^{n}$$

$$\Rightarrow \mathrm{Res}\left(\frac{1}{z^2-4}; 2\right) = \frac{(-1)^{-1+1}}{4^{-1+2}} = \frac{1}{4}$$

NOT simply connected

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7. (5 points) Let $D = \{z \in \mathbb{C} : z \neq 0\}$ be the punctured plane. Is there a function f such that

(i) f is analytic in D

(ii) $e^{f(z)} = z$ for any $z \in D$?

$\log z$

Justify your answer.