Complex Analysis 5/16

IV. $\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$

<u>Example</u>

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \ ?$$

<u>sol</u>

Note that

$$\int_{|z|=1} f(z)\, dz \overset{(*)}{=} \int_0^{2\pi} f(e^{i\theta})\, i e^{i\theta} d\theta$$

And

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_0^{2\pi} \frac{1}{2+\frac{e^{i\theta}+e^{-i\theta}}{2}}\, d\theta$$

$$= \int_0^{2\pi} \frac{2\, e^{i\theta}\, i}{(e^{i\theta})^2 + 4e^{i\theta} + 1} \frac{d\theta}{i}$$

$$\overset{(*)}{=} \int_{|z|=1} \frac{dz}{z^2+4z+1} \cdot \frac{2}{i}$$

Residue Thm

$$= \frac{2}{i}\, 2\pi i\, \operatorname{Res}\left(\frac{1}{z^2+4z+1}; -2+\sqrt{3}\right)$$

$$z^2+4z+1=0 \Rightarrow z = \frac{-4\pm\sqrt{12}}{2} = -2\pm\sqrt{3}$$

$$|-2+\sqrt{3}| < 1, \quad |-2-\sqrt{3}| > 1$$

$$= \frac{2}{i} 2\pi i \left. \frac{1}{(z^2+4z+1)'} \right|_{z=-2+\sqrt{3}}$$

$$= \frac{2}{3}\sqrt{3}\,\pi \qquad \#$$

<u>key points:</u>

① Write $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$, $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$

② change to the form $\displaystyle\int_{|z|=1} f(z)\,dz$

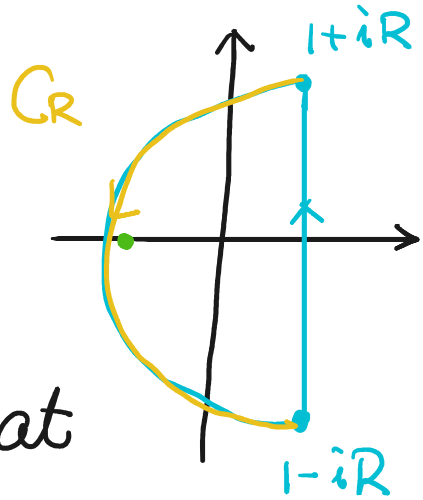
③ Use Residue Thm.

Computation of complex line integral

<u>Example 1</u> (p.161–162)

Let I be the line $z(t) = 1+it$, $-\infty < t < \infty$

$$\int_I \frac{e^z}{(z+2)^3}\,dz = ?$$



<u>sol</u>

Let C_R be the left semicircle of radius $R > 3$ centered at $z = 1$. Then

$$\int_{1-iR}^{1+iR} \frac{e^z}{(z+2)^3}\,dz + \int_{C_R} \frac{e^z}{(z+2)^3}\,dz$$

$$= 2\pi i \, \mathrm{Res}\left(\frac{\quad}{(z+2)^3} ; -2 \right)$$

$$= \frac{1}{(z+2)^3} e^{z+2} e^{-2} = \frac{e^{-2}}{(z+2)^3} \sum_{n=0}^{\infty} \frac{(z+2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{e^{-2}}{n!} (z+2)^{n-3} \qquad n-3 = -1 \Rightarrow n = 2$$

$$= 2\pi i \, \frac{e^{-2}}{2!} = \frac{\pi i}{e^2}$$

Since $|e^z| = e^{\mathrm{Re}\, z} \leq e \quad \forall \, z \in C_R$,

exer $\rightarrow$ $\exists$ constant A s.t. $\quad \left|\frac{1}{(z-1)^3}\right| \left|\frac{(z-1)^3}{(z+2)^3}\right| \rightarrow 1$ as $R \to \infty$

$$\left| \int_{C_R} \frac{e^z}{(z+2)^3} dz \right| \leq \mathrm{length}(C_R) \cdot \sup_{z \in C_R} \left| \frac{e^z}{(z+2)^3} \right|$$

$$\leq \pi R \cdot \frac{A}{R^3} \longrightarrow 0 \quad \text{as } R \to \infty$$

So $$\int_I \frac{e^z}{(z+2)^3} dz = \lim_{R \to \infty} \frac{\pi i}{e^2} = \frac{\pi i}{e^2} \quad \#$$

Example 2 (p 162-163)

$$\int_{|z|=1} \frac{dz}{\sqrt{6z^2-5z+1}} = ?$$

where the square root is $\sqrt{2}$ at $z=1$

sol (sketch)

① (some [illegible] Problem 6) call it $\sqrt{6z^2-5z+1}$

① (See HW3. Problem 6)

$= e^{\frac{1}{2}\log(6z^2-5z+1)}$

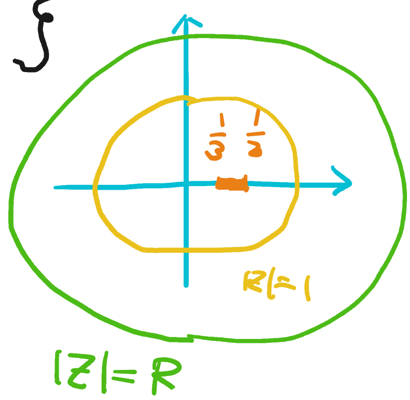
$\exists!$ function $f(z)$ analytic in

$$\mathbb{C} - \{ z \in \mathbb{R} \subseteq \mathbb{C} : \tfrac{1}{3} \leq z \leq \tfrac{1}{2} \}$$

s.t.

$$f(z)^2 = 6z^2 - 5z + 1,$$

$$f(1) = \sqrt{2}$$



② Since $\sqrt{6z^2-5z+1} \sim \sqrt{6}\, z$ for large $|z|$,

exer

$$\int_{|z|=R} \frac{dz}{\sqrt{6z^2-5z+1}} \to \int_{|z|=R} \frac{dz}{\sqrt{6}\, z} = \frac{2\pi i}{\sqrt{6}}$$

as $R \to \infty$

③ By Homotopy Thm,

$$\int_{|z|=1} \frac{dz}{\sqrt{6z^2-5z+1}} = \int_{|z|=R} \frac{dz}{\sqrt{6z^2-5z+1}} \to \frac{2\pi i}{\sqrt{6}}$$

$$\Rightarrow \int_{|z|=1} \frac{dz}{\sqrt{6z^2-5z+1}} = \frac{2\pi i}{\sqrt{6}}$$

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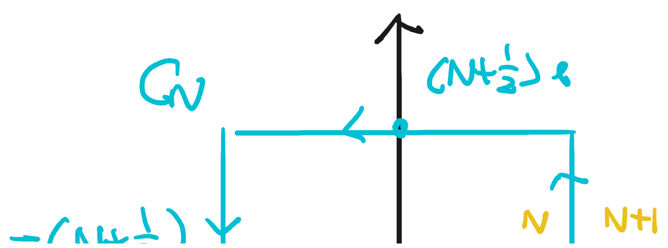
<u>Sum</u> (§11.2)

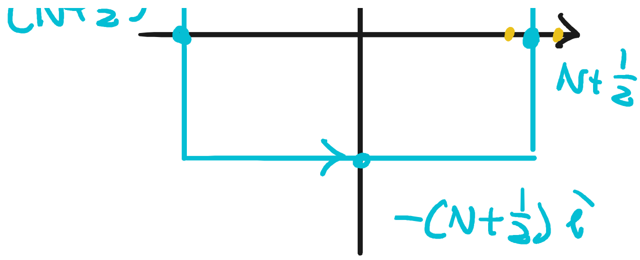
<u>Example</u>

Show $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

pf



Let C_N be the curve



The function

$$\frac{1}{z^2}\pi\cot\pi z = \frac{\pi}{z^2}\frac{\cos\pi z}{\sin\pi z}$$

has poles at $n \in \mathbb{Z}$

exer $|\cot\pi z| < 2$ on C_N

$$\left|\int_{C_N}\frac{1}{z^2}\underbrace{\pi\cot\pi z}_{|\ \ \ |\leq 2\pi}\,dz\right| \leq \mathrm{length}(C_N)\cdot\sup_{C_N}|\cdots|$$

$$\leq (8N+4)\,2\pi\,\underbrace{\max_{z\in C_N}\left|\frac{1}{z^2}\right|}_{\sim\frac{1}{N^2}} \longrightarrow 0 \text{ as } N\to\infty$$

$\Rightarrow$

$$\int_{C_N}\frac{\pi}{z^2}\cot\pi z\,dz$$

$$\left.\frac{(\pi\cos\pi z)/z^2}{(\sin\pi z)'}\right|_{z=n} = \frac{1}{n^2}$$

$\pi\cos\pi z \to (\sin\pi z)'$

$$= 2\pi i\left(\mathrm{Res}\left(\frac{\pi\cot\pi z}{z^2};0\right)+\sum_{\substack{n=-N\\n\neq0}}^{N}\mathrm{Res}\left(\frac{\frac{\pi\cos\pi z}{z^2}}{\sin\pi z};n\right)\right)$$

$$= 2\pi i\left(\mathrm{Res}\left(\frac{\pi\cot\pi z}{z^2};0\right)+2\sum_{n=1}^{N}\frac{1}{n^2}\right)$$

$$\longrightarrow 0 \quad \text{as} \quad N\to\infty$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{1}{2}\mathrm{Res}\left(\frac{\pi\cot\pi z}{z^2};0\right)$$

$$= -\frac{1}{2}\left(-\frac{\pi^2}{3}\right)$$

Fact: Laurent expansion of $\cot z$

is $\frac{1}{z} - \frac{z}{3} - \frac{z^3}{45}\cdots\cdots$

$$= \frac{\pi^2}{6} \qquad \#$$

$\Rightarrow$ Laurent expansion of $\frac{\pi \cot \pi z}{z^2}$ is

$$\frac{\pi}{\pi z^3} - \frac{\pi^2}{3z} - \frac{\pi^4 z}{45} \cdots$$

z^{-1}

$\frac{\pi \cot \pi z}{z^4}$ $\quad z^3$

$$\left(\text{Similarly, } \sum_{n=1}^{\infty} \frac{1}{n^4} = -\frac{1}{2}\left(-\frac{\pi^4}{45}\right) = \frac{\pi^4}{90}\right)$$

Remark

see wiki or Ch 18

- The Riemann zeta function $\zeta(z)$ is a meromorphic function with a simple pole at $z=1$ with residue 1 s.t.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

when RHS makes sense $(\operatorname{Re}(z) > 1)$

Note

$\zeta(2) = \frac{\pi^2}{6}$

$\zeta(4) = \frac{\pi^4}{90}$

- Fact: $\zeta(-2k) = 0 \quad \forall k \in \mathbb{N}$

- Conjecture (Riemann hypothesis):

$$\{\text{zeros of } \zeta(z)\} \setminus \{-2k : k \in \mathbb{N}\} \subseteq \{z \in \mathbb{C} : \operatorname{Re}(z) = \tfrac{1}{2}\}$$

Example

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

pf

Consider same C_N, but a different function

$$\frac{1}{z^2}\pi\csc\pi z = \frac{\pi}{z^2\sin\pi z}$$

A similar argument shows that

$$\frac{\frac{\pi}{z^2}}{(\sin\pi z)'}\Big|_{z=n} = \frac{(-1)^n}{n^2}$$

$$\int_{C_N}\frac{\pi\csc\pi z}{z^2}dz$$

$$= 2\pi i\left(\operatorname{Res}\left(\frac{\pi\csc\pi z}{z^2};0\right) + \sum_{\substack{n=-N\\n\neq 0}}^{N}\underbrace{\operatorname{Res}\left(\frac{\pi\csc\pi z}{z^2};n\right)}_{\shortparallel}\right)$$

$$\longrightarrow 0 \quad \text{as} \quad N\to\infty$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = -\frac{1}{2}\left(\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{(-1)^n}{n^2}\right)$$

$$= \frac{1}{2}\operatorname{Res}\left(\frac{\pi\csc\pi z}{z^2};0\right) = \frac{\pi^2}{12}$$

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Fact:

$$\csc z = \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \cdots$$

Example

$$\sum_{k=0}^{n}\binom{n}{k}^2 = ?$$

sol

Note that

$$\binom{n}{k} = \text{coeff of } z^k \text{ in } (1+z)^n$$

$$\binom{n}{k} = \text{coeff of } z^{-k} \text{ in } \left(1+\frac{1}{z}\right)^n$$

$\binom{n}{k}$ = coeff of z^k in $(1+z)^n$

$$\Rightarrow \sum_{k=0}^{n} \binom{n}{k}^2 = \text{coeff of } z^k \cdot z^{-k} = 1 \text{ in } (1+z)^n \cdot \left(1+\frac{1}{z}\right)^n$$

So

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \frac{1}{2\pi i} \int_{|z|=1} (1+z)^n \left(1+\frac{1}{z}\right)^n \cdot \frac{dz}{z}$$

$$= \frac{1}{2\pi i} \int_{|z|=1} \frac{(1+z)^{2n}}{z^{n+1}} \, dz$$

$$= \text{coeff of } z^n \text{ in } (1+z)^{2n}$$

$$= \binom{2n}{n} \qquad \#$$