Complex Analysis 5/12

Remark

- The last method on Monday works for
$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos x \, dx, \qquad \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin x \, dx$$
when $\deg Q > \deg P$, $Q(x) \neq 0 \ \forall x \in \mathbb{R}$.

- If $Q(x)$ and $\cos x$ (resp. $Q(x)$ and $\sin x$) have a common simple zero, one can modify the method by considering
$$\frac{P(x)}{Q(x)}(e^{iz} \pm i) \qquad \left(\text{resp } \frac{P(x)}{Q(x)}(e^{iz} \pm 1)\right)$$

Example
$$\int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx = ?$$

sol

at $x=0$, $\frac{1}{0}$, NOT good!!

Note that

$$\frac{\sin x}{x} = \operatorname{Im} \frac{e^{ix}}{x} = \operatorname{Im} \frac{e^{ix} - 1}{x}$$

$=0$ at $x=0$

and
$$\frac{e^{iz} - 1}{z} = \frac{1}{z}\left(\sum_{n=0}^{\infty} \frac{1}{n!}(iz)^n - 1\right) \quad \forall z \neq 0$$

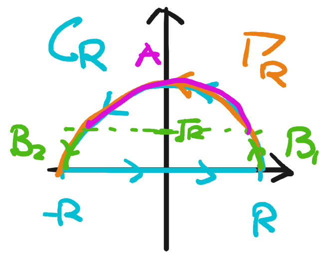
NO $\frac{1}{z^k}$ terms $k > 1$

$$= \sum^{\infty} \frac{(i)^n}{\ } z^{n-1} = i - \frac{z}{\ } + \cdots$$

$\sum_{n=1} \quad n! \quad \ldots \quad z \ldots$

"is" entire (0 is removable singularity)

$$\Rightarrow \int_{C_R} \frac{e^{iz}-1}{z}\,dz = 0$$



$$= \int_{-R}^{R} \frac{e^{ix}-1}{x}\,dx + \int_{\Gamma_R} \frac{e^{iz}-1}{z}\,dz$$

$$\Rightarrow \int_{-R}^{R} \frac{e^{ix}-1}{x}\,dx = -\int_{\Gamma_R} \frac{e^{iz}-1}{z}\,dz$$

$$= \int_{\Gamma_R} \frac{1}{z}\,dz - \int_{\Gamma_R} \frac{e^{iz}}{z}\,dz$$

$\frac{e^{-\operatorname{Im} z}}{R} \le \frac{e^{-\sqrt{R}}}{R}$ $\;\parallel$

Note that $\quad \sqrt{R} = R\sin\varphi \quad \varphi = \sin^{-1}\frac{\sqrt{R}}{R} \quad R\varphi \quad R(\pi - 2\varphi)$

$$\left|\int_A \frac{e^{iz}}{z}\,dz\right| \le \text{length}(A)\cdot \sup_{z\in A}\left|\frac{e^{iz}}{z}\right|$$

$R\sin\varphi = \sqrt{R}$

$\frac{\sqrt{R}}{R\varphi} = \frac{\sin\varphi}{\varphi} \ge \frac{\sin\varphi}{\varphi}\Big|_{\frac{\pi}{2}} = \frac{2}{\pi}$ $\quad 0 \le \varphi \le \frac{\pi}{2}$

$\Rightarrow R\varphi \le \frac{\pi}{2}\sqrt{R}$

$$\le \pi e^{-\sqrt{R}} \longrightarrow 0 \quad \text{as } R \to \infty$$

$R\cdot\varphi \le \frac{\pi}{2}\sqrt{R}$ $\qquad \parallel\; \frac{e^{-\operatorname{Im} z}}{R} \le \frac{1}{R}$

$$\left|\int_{B_1} \frac{e^{iz}}{z}\,dz\right| \le \text{length}(B)\cdot \sup_{z\in B}\left|\frac{e^{iz}}{z}\right|$$

$$\le \frac{\pi}{2}\,\frac{1}{\sqrt{R}} \longrightarrow 0 \quad \text{as } R\to\infty$$

Similarly, $\quad iz \ldots$

$$\left|\int_{B_2} \frac{e}{z}\, dz\right| \longrightarrow 0 \quad \text{as} \quad R\to\infty$$

So $$\int_{\Gamma_R} \frac{e^{iz}}{z}\, dz \longrightarrow 0 \quad \text{as} \quad R\to\infty$$

$\Rightarrow$ $$\int_{-\infty}^{\infty} \frac{e^{ix}-1}{x}\, dx = \lim_{R\to\infty}\int_{-R}^{R}\frac{e^{ix}-1}{x}\, dx$$

$$= \lim_{R\to\infty}\left(\int_{\Gamma_R}\frac{1}{z}\, dz - \underbrace{\int_{\Gamma_R}\frac{e^{iz}}{z}\, dz}_{=0 \text{ as } R\to\infty}\right) = \pi i$$

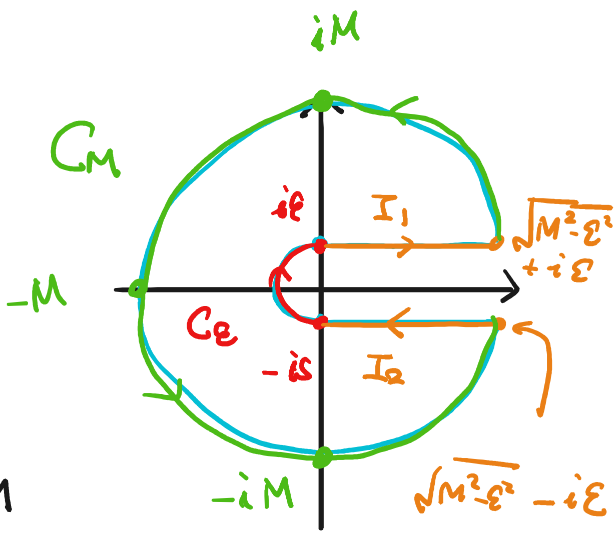
$\Gamma_R : Re^{i\theta}$, $\theta\in[0,\pi]$

$$= \int_0^{\pi} \frac{1}{\cancel{R}e^{i\theta}}\, i\cancel{R}e^{i\theta}\, d\theta = \pi i$$

$\Rightarrow$ $$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \operatorname{Im}\left(\int_{-\infty}^{\infty}\frac{e^{ix}-1}{x}\, dx\right)$$

$$= \operatorname{Im}(\pi i) = \pi. \qquad \#$$

III. $\int_0^{\infty} \frac{P(x)}{Q(x)}\, dx$



Example $$\int_0^{\infty} \frac{dx}{1+x^3} = ?$$

sol

Let
$$K_{\varepsilon,M} = C_\varepsilon \cup I_1 \cup I_2 \cup C_M$$
be the curve as in the diagram

Note that $\log z$ can defined on

$$\underbrace{\mathbb{C} - \mathbb{R}_{\geq 0}}_{} = \mathbb{C} - \{x \in \mathbb{R} : x \geq 0\}$$

$K_{\varepsilon,M} \subseteq$

(Choose $0 < \operatorname{Arg} z < 2\pi$)

Consider

$$\int_{K_{\varepsilon M}} \frac{1}{1+z^3} \log z \, dz$$

① (i) Since $\frac{1}{1+z^3}$ is continuous at $z=0$,

$\exists\ \delta_0 > 0,\ A > 0$ s.t.

$$\left| \frac{1}{1+z^3} \right| \leq A \qquad \forall\ |z| < \delta_0$$

Also note that for $z \in C_\varepsilon$,

$$|\log z| = |\log \varepsilon e^{i\theta}| = |\log \varepsilon + i\underset{\leq 2\pi}{\theta}|$$

$$\leq |\log \varepsilon| + 2\pi$$

$\Rightarrow \forall \varepsilon < \delta_0$,

$$\left| \int_{C_\varepsilon} \frac{\log z}{1+z^3} dz \right| \leq \underbrace{\operatorname{length}(C_\varepsilon)}_{=\pi\varepsilon} \cdot \underbrace{\sup_{z \in C_\varepsilon} \left| \frac{\log z}{1+z^3} \right|}_{\leq A(|\log \varepsilon| + 2\pi)}$$

$$\leq \pi\varepsilon \cdot A \cdot (|\log \varepsilon| + 2\pi) \to 0$$

as $\varepsilon \to 0$

(ii) For $z \in C_M$,

$$\left| \frac{\log z}{1+z^3} \right| \leq \frac{|\log M| + 2\pi}{M^3 - 1}$$

$$\Rightarrow \left| \int_{C_M} \frac{\log z}{z^3+1}\,dz \right| \leq 2\pi M \cdot \frac{|\log M| + 2\pi}{M^3-1} \to 0$$

as $M \to \infty$

(iii)

$$\int_{I_1} \frac{\log z}{z^3+1}\,dz$$

I_1: $i\varepsilon + t$, $0 \leq t \leq \sqrt{M^2-\varepsilon^2}$

$$= \int_0^{\sqrt{M^2-\varepsilon^2}} \frac{\log(i\varepsilon+t)}{(i\varepsilon+t)^3+1}\,dt$$

Im$(\log z)$

↓

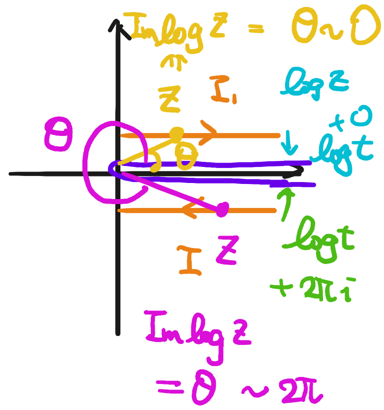
$0 < \operatorname{Arg} z < 2\pi$

$\log z$ on $\mathbb{C} - \mathbb{R}_{\geq 0}$

exer

$$\xrightarrow{\text{as } \varepsilon \to 0} \int_0^M \frac{\log t}{t^3+1}\,dt$$

$$\xrightarrow{\text{as } M \to \infty} \int_0^\infty \frac{\log t}{t^3+1}\,dt$$



But

$$\int_{I_2} \frac{\log z}{z^3+1}\,dz$$

$$\xrightarrow{\text{as } \varepsilon \to 0} -\int_0^M \frac{\log t + 2\pi i}{t^3+1}\,dt$$

$$\xrightarrow{\text{as } M \to \infty} -\int_0^\infty \frac{\log t + 2\pi i}{t^3+1}\,dt$$

Therefore,

$$\int_{K_{M,\varepsilon}} \frac{\log z}{z^3+1}\,dz \longrightarrow -2\pi i \int_0^\infty \frac{1}{t^3+1}\,dt$$

as $M \to \infty$, $\varepsilon \to 0$

② Note that z^3+1 has 3 zeros: $e^{i\frac{\pi}{3}}, e^{i\pi}, e^{i\frac{5\pi}{3}}$

$$\operatorname{Res}\left(\frac{\log z}{1+z^3}; e^{i\frac{\pi}{3}}\right) = \left.\frac{\log z}{3z^2}\right|_{e^{i\frac{\pi}{3}}} = \frac{i\pi}{9}\left(\frac{-1}{2} - \frac{\sqrt{3}}{2} i\right)$$

$\operatorname{Res}(\frac{g}{f}; z_0)$

f has a simple zero at z_0

$g(z_0) \neq 0$

$\Rightarrow \operatorname{Res}(\frac{g}{f}; z_0) = \frac{g(z_0)}{f'(z_0)}$

$$\operatorname{Res}\left(\frac{\log z}{1+z^3}; e^{i\pi}\right) = \left.\frac{\log z}{3z^2}\right|_{e^{i\pi}} = \frac{i\pi}{3}$$

$$\operatorname{Res}\left(\frac{\log z}{1+z^3}; e^{i\frac{5\pi}{3}}\right) = \left.\frac{\log z}{3z^2}\right|_{e^{i\frac{5}{3}\pi}} = \frac{5\pi}{9} i \left(\frac{-1}{2} + \frac{\sqrt{3}}{2} i\right)$$

So, by <u>Residue Thm</u>, for M large, ε small,

$$\int_{K_{M,\varepsilon}} \frac{\log z}{z^3+1} dz = 2\pi i \left(\frac{i\pi}{9}\left(\frac{-1}{2} - \frac{\sqrt{3}}{2} i\right) + \frac{i\pi}{3} + \frac{i5\pi}{9}\left(\frac{-1}{2} + \frac{\sqrt{3}}{2} i\right) \right)$$

$$= -\frac{2\pi}{9}\sqrt{3} \cdot 2\pi i$$

$$\xrightarrow{\text{by } ①} -2\pi i \int_0^\infty \frac{1}{x^3+1} dx$$

$$\Rightarrow \int_0^\infty \frac{1}{x^3+1} dx = \frac{2\pi}{9}\sqrt{3} \quad . \quad \#$$

<u>Remark</u>

- The above method works for

$$\int_a^\infty \frac{P(x)}{Q(x)} dx, \qquad \deg Q - \deg P \geq 2$$

$$Q(x) \neq 0 \quad \forall x \in [a, \infty)$$

Consider $\frac{P(z)}{Q(z)} \log(z-a)$

- This method also applies to $\int_0^\infty \frac{x^{\alpha-1}}{P(x)}\,dx$

  $\deg P \geq 1, \quad 0<\alpha<1, \quad P(x)\neq 0 \quad \forall x\in[0,\infty)$

<u>Example</u> $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \,?$ $\qquad (\alpha=\tfrac{1}{2})$

<u>sol</u>

Let $K_{M,\varepsilon}$ be as in the previous example.

Consider

$$\sqrt{z} = e^{\frac{1}{2}\overbrace{\log z}^{2\pi i \text{ difference}}} \qquad (0<\operatorname{Arg} z<2\pi)$$

is analytic in $\mathbb{C}-\mathbb{R}_{\geq 0}$

Similar as in the previous example, one can show

$$\int_{C_\varepsilon} \frac{dz}{\sqrt{z}(1+z)} \longrightarrow 0 \quad \text{as } \varepsilon\to 0$$

$$\int_{C_M} \frac{dz}{\sqrt{z}\,(1+z)} \longrightarrow 0 \quad \text{as } M\to\infty$$

$$\int_{I_1\cup I_2} \frac{dz}{\sqrt{z}\,(1+z)} \longrightarrow \Big(1-\underbrace{e^{-\frac{1}{2}(2\pi i)}}_{=-1}\Big)\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)}$$

$$= 2\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)}$$

By Residue Thm, for M large, ε small,

$$\int_{K_{\varepsilon,M}} \frac{dz}{\sqrt{z}\,(1+z)} = 2\pi i \,\mathrm{Res}\left(\frac{1}{\sqrt{z}\,(1+z)}\,;\, -1\right)$$

$$= 2\pi i \, \frac{z^{-\frac{1}{2}}}{(1+z)'}\Bigg|_{z=-1} = 2\pi$$

$$\longrightarrow \quad 2\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)}$$

So $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \pi$. #