Complex Analysis 5/9

Recall

- f: analytic in a simply connected domain D
  γ: closed curve in D
  $z_1, \cdots, z_m$: singularities of f in D

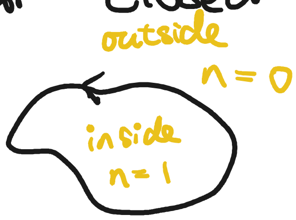
$\Rightarrow$
$$\boxed{\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{m} \underbrace{n(\gamma, z_k)}_{\text{winding number}} \cdot \operatorname{Res}(f; z_k)}$$

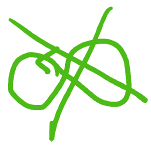
$= \gamma$ 绕 z_k 轉幾圈

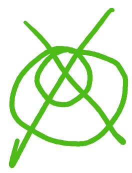
- $p_1, \cdots, p_m$: poles of f
  $z_1, \cdots, z_n$: zeros of f

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n} \operatorname{ord}(z_k)\cdot n(\gamma, z_k) - \sum_{j=1}^{m} \operatorname{ord}(p_j)\cdot n(\gamma, p_j)$$

- regular closed curve:
  e.g.







Rouché Thm (Thm 10.10)

Suppose that f and g are analytic inside and on a regular closed curve γ and that

$\cancel{>0}$ no need ← correction

$$|f(z)| > |g(z)| \quad \forall\, z \in \gamma.$$

$\Rightarrow |f+g| \geq |f|-|g| > 0$ on γ ⊛
and $|f| > 0$ on γ

Then

$$Z_\gamma(f+g) = Z_\gamma(f)$$

where $Z_\gamma(f)$ is the number of zeros of f inside γ, counting multiplicities.

pf

① Note that if $f(z) = A(z)B(z)$, then

$$\frac{f'}{f} = \frac{A'}{A} + \frac{B'}{B}$$

$$\Rightarrow \int_\gamma \frac{f'}{f}\,dz = \int_\gamma \frac{A'}{A}\,dz + \int_\gamma \frac{B'}{B}\,dz.$$

② Since $f+g = f\left(1+\frac{g}{f}\right)$, we have

Argument Principle

$$Z_\gamma(f+g) = \frac{1}{2\pi i}\int_\gamma \frac{(f+g)'}{(f+g)}\,dz \quad (f+g) \neq 0 \text{ on } \gamma \text{ by ⊛}$$

$$\overset{①}{=} \frac{1}{2\pi i}\left(\int_\gamma \frac{f'}{f}\,dz + \int_\gamma \frac{\left(1+\frac{g}{f}\right)'}{\left(1+\frac{g}{f}\right)}\,dz\right)$$

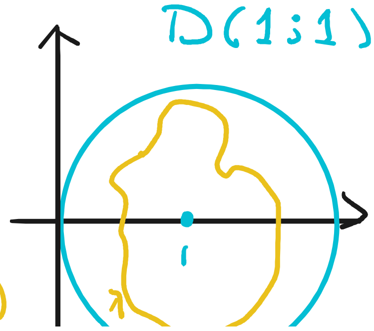
$\neq 0$ on γ by ⊛ $\neq 0$ on γ

$$= Z_\gamma(f) + \frac{1}{2\pi i}\int_\gamma \frac{\left(1+\frac{g}{f}\right)'}{\left(1+\frac{g}{f}\right)}\,dz$$

③ Since $|f(z)| > |g(z)|$ $\forall z \in \gamma$, we have

$$\left|\frac{g}{f}\right| < 1 \quad \text{on } \gamma$$

$$\Rightarrow \left|\left(1+\frac{g}{f}\right) - 1\right| < 1$$



Since $D(1;1)$ is simply connected, by Thm 8.8

$\exists$ analytic branch h of $\log$ in $D(1;1)$

$D(1;1)$

h: analytic in $D(1;1)$

$(e^{h(z)} = z \quad \forall z \in D(1;1))$

$$\Rightarrow \left(h\left(1+\frac{g(z)}{f(z)}\right)\right)' = \frac{\left(1+\frac{g(z)}{f(z)}\right)'}{1+\frac{g(z)}{f(z)}}$$

$$\Rightarrow \int_\gamma \frac{(1+\frac{g}{f})'}{(1+\frac{g}{f})}\,dz = h\left(1+\frac{g}{f}\right)\Big|_{z=\gamma(0)}^{\gamma(1)} = 0$$

$\gamma(1) = \gamma(0)$

So by ②, $Z_\gamma(f+g) = Z_\gamma(f)$. #

<u>Example</u>

Show that the polynomials

$$\underbrace{2z^{10}}_{f} + \underbrace{4z^2}_{g} + \underbrace{1}_{f} \quad \text{and} \quad \underbrace{2z^{10}}_{f} \underbrace{- 4z^2}_{\tilde{g}} + \underbrace{1}_{f}$$

has exactly 2 zeros (counting multiplicities) in $|z| < 1$

pf

Note that on $|z| = 1$ (γ),

$$|4z^2| = 4 > 3 = |2z^{10}| + 1 \geq |2z^{10}+1|$$

By Rouché Thm,

$$Z_\gamma(2z^{10}+4z^2+1) = Z_\gamma(2z^{10}-4z^2+1)$$

$$= Z_\gamma(4z^2) = 2$$

$= \Sigma_r(4\Sigma) \Rightarrow 2$. #

Ch11-12 Applications of Residue Thm

(e.g. computation of integration, Σ)

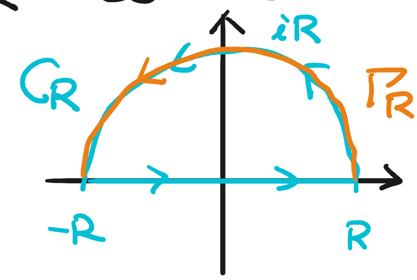
I. $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$:

Example $\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = \ ?$

sol

① Let C_R be the closed curve

Note: $x^4+1 \neq 0 \quad \forall x \in \mathbb{R}$



where R is large enough so that all the zeros of $\underset{\sim}{z^4+1}$ in

no zeros on the real axis

$H^+ = \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$

are inside C_R

Let Γ_R be $Re^{i\theta}$, $\theta \in [0, \pi]$. Then

$$\int_{C_R} \frac{1}{z^4+1}\,dz = \underbrace{\int_{-R}^{R} \frac{1}{x^4+1}\,dx}_{\to \int_{-\infty}^{\infty} \frac{1}{x^4+1}dx \text{ as } R\to\infty} + \int_{\Gamma_R} \frac{1}{z^4+1}\,dz$$

Note that

"M-L inequality"

$|z^4+1| \geq |z|^4 - 1 = R^4 - 1$

$$\left| \int_{\Gamma_R} \frac{1}{z^4+1}\,dz \right| \leq \underbrace{(\pi R)}_{\text{length of}} \cdot \sup_{z \in \Gamma_R} \left| \frac{1}{z^4+1} \right| \leq \frac{1}{R^4-1}$$

$\propto 1/R$

$$= \frac{\pi R}{R^4 - 1} \longrightarrow 0 \quad \text{as } R \to \infty$$

So

$$\lim_{R\to\infty} \int_{C_R} \frac{1}{z^4+1}\,dz = \int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx$$

② Let

$z^4 = -1 = e^{\pi i} \Rightarrow z = e^{\frac{\pi}{4}i} \cdot \bigcirc$

4 roots of $z^4 = 1$

$$Z^+(z^4+1) = \{z \in H^+ : z^4+1 = 0\}$$

$$= \{ e^{\frac{\pi}{4}i}, \; e^{\frac{3}{4}\pi i} \}$$

simple poles of $\frac{1}{z^4+1}$

Recall

$f = \frac{A}{B}$

$A(z_0) \neq 0$

z_0 : simple zero of B

$\Rightarrow \mathrm{Res}(f; z_0) = \frac{A(z_0)}{B'(z_0)}$

Note

- $\mathrm{Res}\left(\frac{1}{z^4+1}; e^{\frac{\pi}{4}i}\right)$

$$= \frac{1}{(z^4+1)'}\Bigg|_{z = e^{\frac{\pi}{4}i}} = \frac{1}{4(e^{\frac{\pi}{4}i})^3}$$

$$= \frac{1}{4} e^{-\frac{3}{4}\pi i} = \frac{-1}{8}(\sqrt{2} + i\sqrt{2})$$

- $\mathrm{Res}\left(\frac{1}{z^4+1}; e^{\frac{3}{4}\pi i}\right) = \frac{1}{4(e^{\frac{3}{4}\pi i})^3} = \frac{1}{8}(\sqrt{2} - i\sqrt{2})$

By Residue Thm,

$$\int_{C_R} \frac{1}{z^4+1}\,dz = 2\pi i\left(\mathrm{Res}\left(\frac{1}{z^4+1}; e^{\frac{\pi}{4}i}\right) + \mathrm{Res}\left(\frac{1}{z^4+1}; e^{\frac{3\pi}{4}i}\right)\right)$$

$$= \frac{\pi i}{4}(-2i\sqrt{2}) = \frac{\sqrt{2}}{2}\pi$$

$\xrightarrow{\text{as } R\to\infty}$ $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = \frac{\sqrt{2}}{2}\pi$

So $$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = \frac{\sqrt{2}}{2}\pi \qquad \#$$

Remark

The method applies to " $\displaystyle\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$ with $\deg Q - \deg P \geq 2$, $Q(x) \neq 0 \ \forall x \in \mathbb{R}$"

The above argument shows that
$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{z \in Z^+(Q)} \operatorname{Res}\left(\frac{P}{Q}; z\right)$$

II. $\int_{-\infty}^{\infty} R(x)\cos x\,dx$ and $\int_{-\infty}^{\infty} R(x)\sin x\,dx$

Example

$$\underbrace{\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx}_{=\operatorname{Re}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}dx\right)} = ? \qquad \underbrace{\int_{-\infty}^{\infty} \frac{\sin x}{x^2+1}\,dx}_{=\operatorname{Im}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}dx\right)} = ?$$

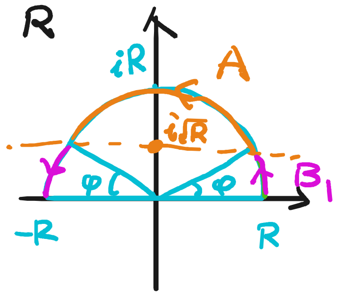
Sol

Let Γ_R : $Re^{i\theta}$, $\theta \in [0, \pi]$

A : $Re^{i\theta}$, $\theta \in [\varphi, \pi-\varphi]$, $\varphi = \sin^{-1}\frac{\sqrt{R}}{R}$

B_1 : $Re^{i\theta}$, $\theta \in [0, \varphi]$,

B_2 : $Re^{i\theta}$, $\theta \in [\pi-\varphi, \pi]$



Consider

$$\int \frac{e^{iz}}{\ \ }\,dz = \int^{R} \frac{\overbrace{e^{ix}}^{=\cos x + i\sin x}}{\ \ }\,dx + \int \frac{e^{iz}}{\ \ }\,dz$$

$$\int_{C_R} \frac{e^{iz}}{z^2+1}\,dz = \int_{-R}^{R} \frac{e^{ix}}{x^2+1}\,dx + \underbrace{\int_{\Gamma_R} \frac{e^{iz}}{z^2+1}\,dz}_{\int_{B_1} + \int_A + \int_{B_2}}$$

$z = x+iy \Rightarrow iz = -y + ix$

$e^{iz} = e^{-y} e^{ix}$

$|e^{iz}| = e^{-\operatorname{Im} z} \leq e^{-\sqrt{R}}$

① (a) $$\left| \int_A \frac{e^{iz}}{z^2+1}\,dz \right| \leq \text{length}(A) \cdot \sup_A \left| \frac{e^{iz}}{z^2+1} \right|$$

$$\leq R(\pi - 2\varphi) \cdot e^{-\sqrt{R}} \frac{1}{R^2-1}$$

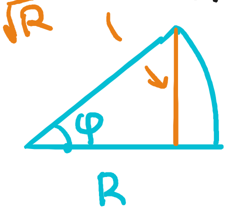
$$= \frac{\pi R}{R^2-1} e^{-\sqrt{R}} \longrightarrow 0 \quad \text{as } R \to \infty$$

$|e^{iz}| = e^{-\operatorname{Im} z} \leq 1$

(b) $$\left| \int_{B_1} \frac{e^{iz}}{z^2+1}\,dz \right| \leq R\cdot\varphi \cdot \frac{1}{R^2-1} \leq \frac{\sqrt{R}}{R^2-1} \cdot \frac{\pi}{2}$$

$\to 0$ as $R \to \infty$

$R\sin\varphi = \sqrt{R}$



$$\frac{\sqrt{R}}{R\varphi} = \frac{\sin\varphi}{\varphi} \geq \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$$

$$\Rightarrow R\varphi \leq \frac{\pi}{2}\sqrt{R}$$

Similarly, $$\int_{B_2} \frac{e^{iz}}{z^2+1}\,dz \longrightarrow 0 \quad \text{as } R \to \infty$$

So $$\int_{\Gamma_R} \frac{e^{iz}}{z^2+1}\,dz \longrightarrow 0 \quad \text{as } R \to \infty$$

$$\int_{C_R} \frac{e^{iz}}{z^2+1}\,dz \longrightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\,dx \quad \text{as } R \to \infty$$

② By Residue Thm, for R large enough,

$$\int_{C_R} \frac{e^{iz}}{z^2+1}\,dz = 2\pi i \operatorname{Res}\left(\frac{e^{iz}}{z^2+1} ; i \right)$$

$$= 2\pi i \cdot \frac{e}{2z}\bigg|_{z=i} = \frac{\pi}{e}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx = \mathrm{Re}\left(\frac{\pi}{e}\right) = \frac{\pi}{e}$$

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+1}\,dx = \mathrm{Im}\left(\frac{\pi}{e}\right) = 0$$

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