Complex Analysis 5/5

Recall

If $f(z) = \sum_{k=-\infty}^{\infty} c_k (z-z_0)^k$ in a deleted nbd of z_0,

$$\mathrm{Res}(f; z_0) := c_{-1}$$

is called the residue of f at z_0

e.g. $\mathrm{Res}\left(\frac{z+1}{z^2}; 0\right) = 1$

$\frac{z+1}{z^2} = \frac{1}{z} + \frac{1}{z^2}$

Remark

(i) If f has has a simple pole at z_0, i.e.,

$$f(z) = \frac{A(z)}{B(z)}$$

where A and B are analytic at z_0, $A(z_0) \neq 0$ and B has a simple zero at z_0 ($B(z_0)=0$, $B'(z_0) \neq 0$)

if $f(z) = \sum_{-1}^{\infty} c_k (z-z_0)^k$, then $\lim \sum_{-1}^{\infty} c_k (z-z_0)^{k+1} = c_{-1}$

then

$$\mathrm{Res}(f; z_0) = \lim_{z \to z_0} (z-z_0) f(z)$$

$$= \lim_{z \to z_0} \frac{A(z)}{\frac{B(z)-B(z_0)}{(z-z_0)}} = \frac{A(z_0)}{B'(z_0)}$$

(ii) If f has a pole of k at z_0,

$$\mathrm{Res}(f; z_0) = \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left((z-z_0)^k f(z) \right) \Big|$$

$\frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left((z-z_0)^k f(z) \right) \Big|_{z=z_0}$

similar as (i)
(see p. 129–130)

(iii) In general, to obtain $\operatorname{Res}(f; z_0)$, one just compute the Laurent expansion of f at z_0

<u>Example</u>

(i) $\operatorname{Res}(\csc z ; 0) = \operatorname{Res}\left(\frac{1}{\sin z} ; 0\right)$ simple pole at 0

$\overset{\text{Remark (i)}}{=} \frac{1}{(\sin z)'}\Big|_{z=0} = \frac{1}{\cos z}\Big|_{z=0} = 1$ #

simple pole at i

(ii) $\operatorname{Res}\left(\frac{1}{z^4-1} ; i\right) = \frac{1}{(z^4-1)'}\Big|_{z=i} = \frac{1}{4(i)^3} = \frac{i}{4}$ #

(iii) $\operatorname{Res}\left(\frac{1}{z^3} ; 0\right) = 0$ #

(iv) $\operatorname{Res}\left(\sin\frac{1}{z-1} ; 1\right) = 1$ because

$$\sin\frac{1}{z-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{z-1}\right)^{2n+1}$$

$$= 1\cdot\frac{1}{z-1} - \frac{1}{3!(z-1)^3} + \frac{1}{5!(z-1)^5} - \cdots$$ #

(v) $\operatorname{Res}\left(\sin\frac{1}{z-1} ; i\right) = 0$ because $\sin\frac{1}{z-1}$ is analytic at i. #

<u>Winding number</u>

<u>Def 10.2</u>

Suppose γ is a closed curve and $a \notin \gamma$.
Then

$$n(\gamma, a) := \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}$$

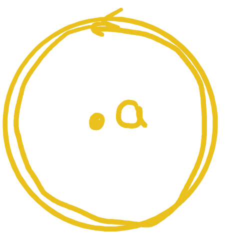
is called the winding number of γ around a

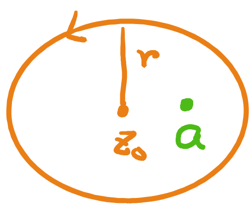
Example

① If $\gamma(\theta) = z_0 + re^{i\theta}$, $0 \le \theta \le 2\pi$, then

$$n(\gamma, a) = \begin{cases} 1 & \text{if } a \in D(z_0; r) \\ 0 & \text{if } a \notin \overline{D(z_0; r)} \end{cases}$$

k times





② If $\gamma(\theta) = a + re^{ik\theta}$, $0 \le \theta \le 2\pi$, then

$$n(\gamma, a) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{1}{re^{ik\theta}} r \cdot ik\, e^{ik\theta} d\theta$$

$$= k$$

Thm 10.3

For any closed curved γ, $a \notin \gamma$,
$n(\gamma, a)$ is an an integer

pf

Suppose γ is given $z(t)$, $0 \le t \le 1$, and

$$F(s) = \int_0^s \frac{z'(t)}{z(t) - a} dt, \quad 0 \le s \le 1$$

$(F(1) = 2\pi i \cdot n(\gamma, a))$

$$\Rightarrow F'(s) = \frac{z'(s)}{z(s)-a} \qquad -z'(s)$$

$$\Rightarrow \frac{d}{ds}\left[(z(s)-a)\cdot e^{-F(s)}\right] = z'(s)e^{-F(s)} + (z(s)-a)(-F'(s))e^{-F(s)}$$

$$= 0$$

$$\Rightarrow (z(s)-a)\cdot e^{-F(s)} = \text{constant} = (z(0)-a)e^{-F(0)}$$

$$= z(0)-a$$

$$\Rightarrow e^{F(s)} = \frac{z(s)-a}{z(0)-a}$$

Since γ is closed, $z(0)=z(1)$, and

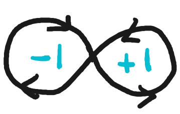
$$e^{F(1)} = \frac{z(1)-a}{z(0)-a} = 1 \qquad e^{x+iy} = e^{x}(\cos y + i\sin y) = 1 \Rightarrow x=0,\ y=2k\pi$$

$$\Rightarrow F(1) = 2\pi k i \text{ for some } k \in \mathbb{Z}$$

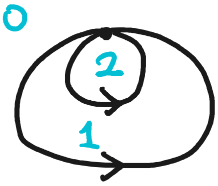
$$\Rightarrow n(\gamma, a) = \frac{1}{2\pi i}F(1) = k \in \mathbb{Z}. \quad \#$$

Example

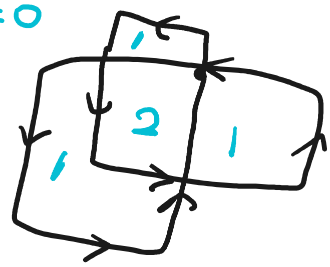
① $n=0$



② $n=0$



③ $n=0$



Residue Thm and applications

Thm 10.5 (Cauchy's Residue Thm)

Suppose f is analytic in a simply connected domain D except for isolated singularities at $z_1, z_2, \cdots, z_m$.

Let γ be a closed piecewise C^1 curve not intersecting any of the singularities.

Then

$$\boxed{\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{m} n(\gamma, z_k)\cdot \operatorname{Res}(f; z_k)}$$

pf

Let $P_k\left(\frac{1}{z-z_k}\right)$ be the principal part of the Laurent expansion at z_k.

$$\Rightarrow \quad g(z) := \begin{cases} f(z) - P_1\left(\frac{1}{z-z_1}\right) - \cdots - P_k\left(\frac{1}{z-z_m}\right) & \text{if } z \neq z_1, \dots, z_m \\ \lim\limits_{z \to z_k} g(z) & \text{if } z = z_k \end{cases}$$

is an analytic function in D

By <u>Closed Curve Thm</u> (Thm 8.6), use the assumption D is simply connected

$$\int_\gamma g(z)\,dz = 0$$

$$\Rightarrow \int_\gamma f(z)\,dz = \sum_{k=1}^{m} \int_\gamma P_k\left(\frac{1}{z-z_k}\right) dz$$

Furthermore, if

$$P_k\left(\frac{1}{z-z_k}\right) = \frac{C_{-1}}{z-z_k} + \frac{C_{-2}}{(z-z_k)^2} + \cdots$$

$(j \neq 1)$

$$\int_\gamma \frac{1}{(z-z_k)^j}\,dz = \left.\frac{(z-z_k)^{1-j}}{1-j}\right|_{z=\gamma(0)}^{\gamma(1)} = 0$$

then

$$\int_\gamma P_k\left(\frac{1}{z-z_k}\right) dz = C_{-1}\int_\gamma \frac{1}{z-z_k}\,dz$$

$$\overset{\text{by def}}{=} \operatorname{Res}(f; z_k)\cdot 2\pi i \cdot n(\gamma, z_k)$$

$$\Rightarrow \int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{m} n(\gamma, z_k)\cdot \mathrm{Res}(f; z_k)$$ #

Example

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{z}\right)^{2n+1} = \frac{1}{z} + \cdots$$

$$\int_{|z|=1} \sin\frac{1}{z}\,dz = 2\pi i\cdot 1\cdot \mathrm{Res}\left(\sin\frac{1}{z}; 0\right)$$

$$= 2\pi i.$$ #

Def 10.7

(holomorphic $\overset{\text{basically}}{=}$ analytic)

We say f is meromorphic in a domain D if f is analytic in D except at isolated poles.

Argument Principle (Thm 10.8, Cor 10.9)

Let f be a meromorphic function in a simply connected domain D with poles $p_1, \cdots, p_m$ and zeros $z_1, \cdots, z_n$. Suppose

$\mathrm{ord}(p_j)$ = the order of the pole p_j

$\mathrm{ord}(z_k)$ = the order of the zero z_k

If γ is a closed piecewise C^1 curve in D, not passing through $p_1, \cdots, p_m, z_1, \cdots, z_n$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n} \mathrm{ord}(z_k)\cdot n(\gamma; z_k) - \sum_{j=1}^{m} \mathrm{ord}(p_j)\cdot n(\gamma; p_j)$$

pf

① If f has a zero of order k at $z=a$, then

$$f(z) = (z-a)^k g(z)$$

where $g(a) \neq 0$, so

$$f'(z) = k(z-a)^{k-1} g(z) + (z-a)^k g'(z)$$

$$\Rightarrow \frac{f'(z)}{f(z)} = \frac{k(z-a)^{k-1} g(z) + (z-a)^k g'(z)}{(z-a)^k \, g(z)}$$

ord(a)

$$= \frac{k}{z-a} + \frac{g'(z)}{g(z)}$$

analytic at $z=a$.

has a simple pole at a with residue k

② Similarly, if f has a pole of order ℓ at $z=b$, then

$$f(z) = (z-b)^{-\ell} h(z) \qquad h(b) \neq 0$$

ord(b)

analytic at b

$$\Rightarrow \frac{f'(z)}{f(z)} = -\frac{\ell}{z-b} + \frac{h'(z)}{h(z)}$$

has a simple pole at b with residue $-\ell$

③ By Residue Thm,

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{n} \mathrm{ord}(z_k) \cdot n(\gamma; z_k)$$

$$- \sum^{m} \mathrm{ord}(p_j)\, n(\gamma; p_j)$$

$j=1$

#

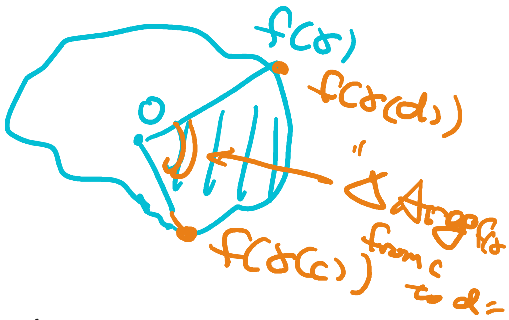
Remark (p136)

The above theorem is known as "Argument Principle" because

$(\log f(z))'$ recall: $\log e^{i\theta} = i\theta$ $\quad \mathrm{Arg}(e^{i\theta})$

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \text{“}\frac{1}{2\pi i}\left(\log f(\gamma(b)) - \log f(\gamma(a))\right)\text{”}$$

$$= \text{“}\frac{1}{2\pi}\,\Delta\,\mathrm{Arg}\, f(\gamma)\text{”}$$



Def 1.24

γ is called a regular closed curve if γ is a simple closed piecewise C^1 curve with $n(\gamma, a) = 0$ or $n(\gamma, a) = 1$ $\forall a \notin \gamma$.

In this case, we will call

$\{a \in \mathbb{C} : n(\gamma, a) = 1\}$ = the inside of γ

$\{a \in \mathbb{C} : n(\gamma, a) = 0\}$ = the outside of γ

e.g.