Complex Analysis 4/28

Recall

- Laurent expansion : $f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$

  (or $f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$)

- Proof of "analytic $\Rightarrow \exists$ Taylor expansion" (Thm 5.5, 6.5, 6.6)

  ① Cauchy Integral Formula (Thm 5.3, Thm 6.4):

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z}\, dw$$

C ← a circle　　need "$(z-z_0)^k$"

  ② Expand $\frac{1}{w-z}$ and integrate

- On Monday, we were proving

Thm 9.9

If f is analytic in

$$A = \{ z \in \mathbb{C} : R_1 < |z| < R_2 \}$$

then f has a Laurent expansion:

$\exists\, a_k \in \mathbb{C}$ s.t.

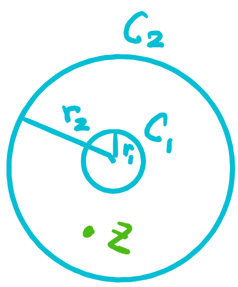
$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k \qquad \forall z \in A.$$

pf

① (Done on Moday)

Let $R_1 < r_1 < r_2 < R_2$.

Fix z, $r_1 < |z| < r_2$



Then

$$\circledast \quad \boxed{f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(\omega)}{\omega - z}\, d\omega - \frac{1}{2\pi i}\int_{C_1} \frac{f(\omega)}{\omega - z}\, d\omega}$$

② (a) For $\omega \in C_2$, since $|\omega| > |z|$,

$$\frac{1}{\omega - z} = \frac{1}{\omega\left(1 - \frac{z}{\omega}\right)} \qquad \left|\frac{z}{\omega}\right| < 1$$

$$= \frac{1}{\omega}\sum_{k=0}^{\infty}\left(\frac{z}{\omega}\right)^k \qquad \left|\frac{z}{\omega}\right| = \frac{|z|}{r_2} \text{ indep of } \omega \quad (\omega \in C_2)$$

Converges uniformly on C_2

(b) For $\omega \in C_1$, since $|\omega| < |z|$,

$$\frac{1}{\omega - z} = -\frac{1}{z\left(1 - \frac{\omega}{z}\right)} \qquad \left|\frac{\omega}{z}\right| < 1$$

$$= -\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{\omega}{z}\right)^k \qquad \left|\frac{\omega}{z}\right| = \frac{r_1}{|z|} \text{ indep of } \omega$$

Converges uniformly on C_1

③ Plug (a), (b) into $\circledast$:

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \sum_{k=0}^{\infty} \frac{f(\omega)}{\omega^{k+1}} z^k \, d\omega$$

$$+ \frac{1}{2\pi i} \int_{C_1} \underbrace{\boxed{\sum_{k=0}^{\infty} f(w) \cdot w^k \cdot \left(\frac{1}{z}\right)^{k+1}}}_{= \sum_{k=-1}^{-\infty} \frac{f(w)}{w^{k+1}} z^k} dw$$

by uniform convergence

$$= \sum_{k=0}^{\infty} \underbrace{\left(\frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{w^{k+1}} dw \right)}_{a_k} \cdot z^k$$

$$+ \sum_{k=-1}^{-\infty} \underbrace{\left(\frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w^{k+1}} dw \right)}_{a_k} \cdot z^k$$

<u>Note</u>: since $\frac{f(w)}{w^{k+1}}$ is analytic in A,

by <u>Homotopy Thm</u>,

$$a_k = \frac{1}{2\pi i} \int_C \frac{f(w)}{w^{k+1}} dw$$

where C is any circle centered at 0 in A

$\Rightarrow$ a_k are independent of r_1, r_2, z. #

<u>Prop</u> (p. 123)

The Laurent expansion is unique.

More explicitly, if $f(z) = \sum^{\infty} a_k z^k$

[illegible] $k=-\infty$

in A, then

$$a_k = \frac{1}{2\pi i}\int_C \frac{f(\omega)}{\omega^{k+1}}\,d\omega$$

circle in A, centered at 0

Cor 9.10

If f is analytic in the annulus $R_1 < |z-z_0| < R_2$, then f has a unique Laurent expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$$

where

$$a_k = \frac{1}{2\pi i}\int_C \frac{f(\omega)}{(\omega-z_0)^{k+1}}\,d\omega$$

and C is any circle $z_0 + re^{i\theta}$, $0\le\theta\le 2\pi$, $R_1 < r < R_2$

(pf: Apply Thm 9.9 and Prop to $g(z) = f(z+z_0)$)

Cor 9.11

If f has an isolated singularity at z_0, then $\exists\ \delta>0$ s.t.

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k, \qquad 0<|z-z_0|$$

$k=-\infty$ $\qquad$ <0

where a_k are defined as in <u>Cor 9.10</u>

(pf : Apply <u>Cor 9.10</u> to the case $R_1 = 0$)

<u>Example</u> (p.124)

Laurent expansion about 0

(i) $$\frac{(z+1)^2}{z} = \frac{z^2+2z+1}{z} = \frac{1}{z} + 2 + z \quad \forall z \neq 0$$

(ii) $$\frac{1}{z^2(1-z)} = \frac{1}{z^2}\cdot\frac{1}{1-z}$$

$$= \frac{1}{z^2}\sum_{k=0}^{\infty} z^k = \sum_{k=0}^{\infty} z^{k-2}$$

Laurent expansion about 0 $\rightarrow$ $$= \sum_{k=-2}^{\infty} z^k \quad \text{for } 0<|z|<1$$

(iii) $$\frac{1}{z^2(1-z)} = \frac{-1}{[1+(z-1)]^2(z-1)}$$

$$= \frac{1}{z-1}\cdot\underbrace{\left(\frac{1}{1+(z-1)}\right)'}$$

$$= \frac{1}{z-1}\cdot\left(\sum_{k=0}^{\infty}(-(z-1))^k\right)'$$

$$= \cancel{\frac{1}{z-1}}\cdot\sum_{k=0}^{\infty}(-1)^k\cdot k\,(z-1)^{\cancel{k-1}\;k-2}$$

$$= \sum_{k=-1}^{\infty} (-1)^k (k+2) (z-1)^k$$

Laurent expansion about 1 for $0 < |z-1| < 1$

(iv) $\exp(\frac{1}{z}) = 1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots$ for $z \neq 0$

Laurent expansion about 0

Laurent expansion and singularity

Def 9.12

If $f(z) = \sum a_k (z-z_0)^k$ is the Laurent expansion of f about an isolated singularity z_0, $\sum_{k=-\infty}^{-1} a_k (z-z_0)^k$ is called the principal part of f at z_0; $\sum_{k=0}^{\infty} a_k (z-z_0)^k$ is called the analytic part.

Remark

(i) If f has a removable singularity at z_0, then all the coefficients a_{-k}, $k>0$, of its Laurent expansion about z_0 are 0.

e.g. $$\frac{\sin z}{z} = \frac{1}{\cancel{z}} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} z^{2n+\cancel{1}}$$

removable singularity at 0

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n}$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

(ii) If f has a pole of order k at z_0, then $a_{-k} \neq 0$ and $a_{-N} = 0 \ \forall N > k$

pf: exercise. See p.125.

e.g. $$\frac{z+1}{z^2} = \frac{1}{z^2} + \frac{1}{z}$$ pole of order 2

(iii) If f has an essential singularity at z_0, then it must have infinitely many nonzero terms in its principal part

e.g. $$\exp\left(\frac{1}{z}\right) = 1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots + \frac{1}{n!\,z^n} + \cdots$$

<u>Cor 9.13</u> (Partial Fraction Decomposition of Rational Functions)

Suppose

$P(z)$ $P(z)$

$$R(z) = \frac{P(z)}{Q(z)} = \frac{\quad}{(z-z_1)^{k_1}(z-z_2)^{k_2}\cdots(z-z_n)^{k_n}}$$

where $P(z)$ is a polynomial with

$$\deg P(z) < \deg Q(z)$$

$\Rightarrow R(z) \longrightarrow 0$ as $z \to \infty$

e.g.

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)}$$

$$= \frac{1}{2}\left(\frac{1}{z-1} + \frac{1}{z+1}\right)$$

$$Q(z) = (z-z_1)^{k_1} \cdots (z-z_n)^{k_n}.$$

$z_1, \ldots, z_n$ are distinct

Then $R(z)$ can be expanded as a sum polynomials in $\frac{1}{z-z_k}$, $k = 1, \ldots, n$

pf

① Since R has a pole of order at most k_1 at z_1

$\Rightarrow$ polynomial of deg at most k_1

$$R(z) = P_1\left(\frac{1}{z-z_1}\right) + A_1(z)$$

where $P_1\left(\frac{1}{z-z_1}\right)$ is the principal part

$A_1(z)$ is the analytic part of R

$\Rightarrow A_1(z) = R(z) - P_1\left(\frac{1}{z-z_1}\right)$ has a removable singularity at z_1 and has the same principal parts as R at $z_2, \ldots, z_n$

analytic at

② If we take $P_k\left(\frac{1}{z-z_k}\right)$ to be the principal part of R around z

principal part of R around z_2.

Repeat ① inductively, we find

as $z \to \infty$ $\to 0$

$$A_n(z) = R(z) - \left[P_1\left(\frac{1}{z-z_1}\right) + \cdots + P_n\left(\frac{1}{z-z_n}\right)\right]$$

"is" an entire function

$P_1(0) + \cdots + P_n(0) = 0$

Furthermore, $A_n \to 0$ as $z \to \infty$

$\Rightarrow$ $A_n(z)$ is bounded

By Liouville Thm (Thm 5.10),

$$A_n = \text{constant} = \lim_{z \to \infty} A_n(z) = 0$$

$\Rightarrow$

$$R(z) = P_1\left(\frac{1}{z-z_1}\right) + \cdots + P_n\left(\frac{1}{z-z_n}\right). \quad \#$$

Ch 10 Residues

Def 10.1

If $f(z) = \sum_{k=-\infty}^{\infty} c_k (z-z_0)^k$ in a deleted nbd of z_0, c_{-1} is called the residue of f at z_0.

Notation: $\mathrm{Res}(f; z_0) \doteq c_{-1}$

Notation: $\operatorname{Res}(f; z_0) := c_{-1}$

e.g. $\operatorname{Res}\left(\frac{z+1}{z^2}; 0\right) = 1 \qquad \left(\frac{z+1}{z^2} = \frac{1}{z} + \frac{1}{z^2}\right)$

$\operatorname{Res}\left(\exp\left(\frac{1}{z}\right); 0\right) = 1 \qquad \left(e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots\right)$