Complex Analysis 4/25

Recall

- z_0 is removable singularity of f if
  $\exists$ g analytic at z_0 s.t. $f \equiv g$ in
  a deleted nbd of z_0

  e.g. $f(z) = \begin{cases} \sin z & , z \neq 2 \\ 0 & , z = 2 \end{cases}$

- pole of order k: $f = \frac{A}{B}$,
  $A(z_0) \neq 0$, B has zero at z_0 of order k

  e.g. $g(z) = \frac{z}{(z-3)^2}$ has a pole of order 2 at 3

- essential singularity: neither removable singularity nor pole

  e.g. $\underbrace{\exp(1/z)}_{= e^{\operatorname{Re}\frac{1}{z}}(\cos(\operatorname{Im}\frac{1}{z}) + i\sin(\operatorname{Im}\frac{1}{z}))}$ has an essential singularity at 0.
  $\Rightarrow$ range $= \mathbb{C} \setminus \{0\}$

Removable singularity

Thm 9.3 (Rieman's Principle of Removable Singularity)
If f has an isolated singularity at z_0 and if $\lim_{z \to z_0} (z - z_0) f(z) = 0$, then the

singularity is removable (*)

pf

Let

$$h(z) = \begin{cases} (z-z_0)f(z) & z \neq z_0 \quad (**) \\ 0 & z = z_0 \end{cases}$$

(*) $\Rightarrow$ h is continuous at z_0

(**) $\Rightarrow$ h is analytic in a deleted nbd of z_0

So h is continuous, analytic except a point in a nbd z_0 (Thm 7.7)

$\Rightarrow$ h is analytic at z_0

$\Rightarrow$

$$g(z) = \begin{cases} \left[\dfrac{h(z) - h(z_0)}{z - z_0}\right] = f(z) \quad (h(z_0) = 0) & z \neq z_0 \\ h'(z_0) & z = z_0 \end{cases}$$

is also analytic at z_0, and

$g = f$ in a deleted nbd of z_0 #

Cor 9.4

If f is bounded in a deleted nbd of an isolated singularity, then the singularity is removable. (bdd $\Rightarrow$ $(z-z_0)f(z) \to 0$ as $z \to z_0$)

Pole

Thm 9.5

If f is analytic in a deleted nbd of z_0 and if $\exists\, k \in \mathbb{N}$ s.t.

$$\lim_{z \to z_0} (z-z_0)^k f(z) \neq 0 \quad \text{but}$$

$$\lim_{z \to z_0} (z-z_0)^{k+1} f(z) = 0,$$

then f has a pole of order k at z_0.

pf

If we set

$$g(z) = \begin{cases} (z-z_0)^{k+1} f(z), & z \neq z_0 \\ 0, & z = z_0 \end{cases}$$

then g is analytic at z_0 (Thm 7.7).

Furthermore, since $g(z_0) = 0$,

$$A(z) = \begin{cases} \dfrac{g(z)}{z-z_0} = (z-z_0)^k f(z), & z \neq z_0 \\ g'(z_0) \neq 0 \text{ by assumption}, & z = z_0 \end{cases}$$

is analytic at z_0.

Note that

$$f(z) = \frac{A(z)}{(z-z_0)^k} \qquad z \neq z_0$$

$A(z_0) \neq 0$

$\Rightarrow$ f has a pole of order k at z_0

#

Remark

By Thm 9.3, Thm 9.3, " $\nexists$ pole of order $\frac{8}{p} \in \mathbb{Q} \setminus \mathbb{Z}$ "

For example,

① if $|f(z)| \le 1/\sqrt{|z|}$ in a deleted nbd of 0, then $|z f(z)| \le |z| \frac{1}{\sqrt{|z|}} = \sqrt{|z|} \to 0$ as $z \to 0$

$\Rightarrow$ by Thm 9.3, 0 is removable

($\nexists$ pole of order $\frac{1}{2}$)

② if $|f(z)| \le \frac{1}{|z|^{5/2}}$ in a deleted nbd of 0

then $|z^2 f(z)| \le \frac{1}{\sqrt{|z|}}$

$\Rightarrow z^2 f(z) = A(z)$ in a deleted nbd of 0

$\Rightarrow f(z) = \frac{A(z)}{z^2}$ has a pole at 0 of order at most 2.

($\nexists$ pole of order $\frac{5}{2}$)

Essential singularity

By Thm 9.3 and Thm 9.5, if f has an essential singularity at z_0, then $\forall N \in \mathbb{N}$

$$(z-z_0)^N f(z) \not\to 0 \quad \text{as } z \to z_0$$

However, $f(z) \not\to \infty$ as $z \to z_0$.

In fast, we have

<u>Casorati-Weierstrass Thm</u> (Thm 9.6)

If f is analytic in a deleted nbd D of z_0 and has an essential singularity at z_0, then the range $R = f(D)$ is dense in $\mathbb{C}$.

pf

Assume R is not dense.

Then $\exists\ \omega \in \mathbb{C}$, $\delta > 0$ s.t.

$$D(\omega; \delta) \cap \underbrace{R}_{= f(D)} = \phi$$

i.e. $$|f(z) - \omega| \geq \delta \quad \forall z \in D$$

$$\Rightarrow \left| \underbrace{\frac{1}{f(z)-\omega}}_{\text{bdd}} \right| \leq \frac{1}{\delta} \quad \forall z \in D$$

By <u>Cor 9.4</u>, $\dfrac{1}{f(z)-\omega}$ has (at most) a removable singularity at $z = z_0$

$\Rightarrow \exists g$ analytic in $D \cup \{z_0\}$ s.t.

$$g(z) = \frac{1}{f(z)-\omega} \quad \forall z \neq z_0$$

$$\Rightarrow \quad f(z) = \omega + \frac{1}{g(z)} = \frac{(\omega g(z)+1)}{g(z)}$$

$\Rightarrow$ f has either a pole (if $g(z_0) = 0$)
or a removable singularity (if $g(z_0) \neq 0$)
at z_0 $\quad (\longrightarrow\!\!\leftarrow)$ $\quad \#$

<u>Remark</u>

In fact, there is a stronger thm (Great Picard's Thm): $f(D) = R = \mathbb{C}$ or $\mathbb{C} \setminus \{\text{a point}\}$

<u>Laurent expansions</u>

<u>Def 9.7</u>

We say $\sum_{k=-\infty}^{\infty} \mu_k = L$ if both $\sum_{k=0}^{\infty} \mu_k$ and $\sum_{k=1}^{\infty} \mu_{-k}$ converge and $\sum_{k=0}^{\infty} \mu_k + \sum_{k=1}^{\infty} \mu_{-k} = L$

We say $\sum_{k=-\infty}^{\infty} \mu_k$ is convergent (resp. converges uniformly) if $\sum_{k=0}^{\infty} \mu_k$ and $\sum_{k=1}^{\infty} \mu_{-k}$ are convergent (resp. converge uniformly)

<u>Thm 9.8</u>

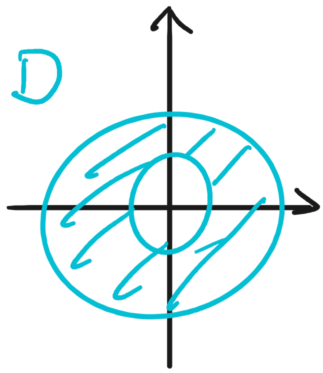
$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ is convergent and analytic in the domain

$$D = \{ z \in \mathbb{C} : \ R_1 < |z| < R_2 \}$$

where

$$R_1 = \overline{\lim_{k\to\infty}} |a_{-k}|^{1/k}$$

$$R_2 = 1 \Big/ \overline{\lim_{k\to\infty}} |a_k|^{1/k}$$



pf

series	domain of convergence
$f_1(z) = \sum_{k=0}^{\infty} a_k z^k$	$\lvert z\rvert < 1 \Big/ \overline{\lim_{k\to\infty}} \lvert a_k\rvert^{1/k} = R_2$
$f_2(z) = \sum_{k=-\infty}^{-1} a_k z^k = \sum_{k=1}^{\infty} a_{-k} \left(\frac{1}{z}\right)^k$	$\left\lvert \frac{1}{z} \right\rvert < 1 \Big/ \overline{\lim_{k\to\infty}} \lvert a_{-k}\rvert^{1/k}$ $\Leftrightarrow$ $\lvert z\rvert > \overline{\lim_{k\to\infty}} \lvert a_{-k}\rvert^{1/k} = R_1$

So, by Thm 2.9, $f(z) = f_1(z) + f_2(z)$ is convergent and analytic in D. ※

Thm 9.9

If f is analytic in the annulus

$$A = \{ z \in \mathbb{C} : R_1 < |z| < R_2 \},$$

then f has a Laurent expansion, i.e.

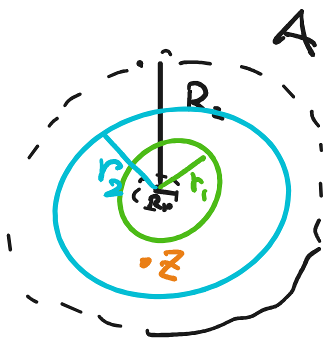
$\exists\ a_k \in \mathbb{C}$ s.t.

$$f(z) = \sum^{\infty} a_k z^k \qquad \forall z \in A$$

pf

① Let $C_j : r_j e^{i\theta}$, $\theta \in [0, 2\pi]$, $j = 1, 2$

$R_1 < r_1 < r_2 < R_2$



Fix z with $r_1 < |z| < r_2$. Then

$$g(\omega) = \frac{f(\omega) - f(z)}{\omega - z} \qquad (g(z) := f'(z))$$

is analytic in A, and by Homotopy Thm (also see Example 2, p113),

$$\int_{C_2 - C_1} g(\omega)\, d\omega = 0$$

$$\Rightarrow \int_{C_2 - C_1} \frac{f(\omega)}{\omega - z}\, d\omega = \int_{C_2 - C_1} \frac{f(z)}{\omega - z}\, d\omega$$

Note that

$$\int_{C_2} \frac{1}{\omega - z}\, d\omega = 2\pi i \quad \text{(Lemma 5.4)}$$

$$\int_{C_1} \frac{1}{\omega - z}\, d\omega = 0$$

$$\Rightarrow \int_{C_2 - C_1} \frac{f(z)}{\omega - z}\, d\omega = 2\pi i f(z)$$

$$\Rightarrow \boxed{f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(\omega)}{\omega - z}\, d\omega - \frac{1}{2\pi i} \int_{C_1} \frac{f(\omega)}{\omega - z}\, d\omega}$$

c.f. Cauchy Integral Formula, p61

Complex Analysis — Homework 6

1. Suppose that f is entire and that $|f(z)| \geq |z|^N$ for sufficiently large z. Show that f must be a polynomial of degree at least N.

2. Find the maximum and minimum moduli of $z^2 - z$ in the disc: $|z| \leq 1$.

3. Show that if f is analytic and nonconstant on a compact set in $\mathbb{C}$, then $\operatorname{Re} f$ and $\operatorname{Im} f$ assume their maxima and minima on the boundary.

4. Let $D = D(0;1)$ be the unit disc and $S^1 = \partial D$ be its boundary. Suppose f is nonconstant and analytic in D and continuous in its closure $\overline{D}$. Show that if $f(S^1) \subset S^1$, then $f(D) = D$.

5. Suppose f is entire and $|f| = 1$ on $|z| = 1$. Prove that there exists $c \in \mathbb{C}$ such that $f(z) = cz^n$ for all $z \in \mathbb{C}$.

6. Suppose that f is analytic in the annulus: $1 \leq |z| \leq 2$, that $|f| \leq 1$ for $|z| = 1$ and that $|f| \leq 4$ for $|z| = 2$. Prove $|f(z)| \leq |z|^2$ throughout the annulus.

7. (a) Suppose that f is analytic and bounded by 1 in the unit disc with $f(\alpha) \neq 0$ for some $|\alpha| < 1$. Show that there exists a function g , analytic and bounded by 1 in the unit disc, with $|g'(\alpha)| > |f'(\alpha)|$.

   (b) Find $\max_f |f'(\alpha)|$ where f ranges over the class of analytic functions bounded by 1 in the unit disc, and α is a fixed point with $|\alpha| < 1$.

8. Let
$$f(z) = \int_0^1 \frac{\sin zt}{t}\, dt.$$
Show that
   (a) f is entire;
   (b) $f'(z) = \displaystyle\int_0^1 \cos zt\, dt.$

9. Given an entire function which is real on the real axis and imaginary on the imaginary axis, prove that it is an odd function, i.e., $f(z) = -f(-z)$.

10. Suppose f is analytic in $|z| < 1, \operatorname{Im} z > 0$, continuous on $|z| \leq 1, \operatorname{Im} z > 0$ and real on the semi-circle: $|z| = 1$, $\operatorname{Im} z > 0$. Show that if we set
$$g(z) = \begin{cases} f(z), & |z| \leq 1, \operatorname{Im} z > 0, \\ \overline{f(1/\overline{z})}, & |z| > 1, \operatorname{Im} z > 0, \end{cases}$$
then g is analytic in the upper half plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

Complex logarithm

Let $D = \{z \in \mathbb{C} : z \neq 0\}$ be the punctured plane. There is NO function f satisfying the conditions

(i) f is analytic in D,

(ii) $e^{f(z)} = z$ for any $z \in D$.

" $\nexists$ log on D " $D = \mathbb{C} \setminus \{0\}$

Proof. Suppose f is a function which satisfies (i) and (ii). Then for each $\theta \in [0, 2\pi]$,

$$e^{f(e^{i\theta})} = e^{u(e^{i\theta})} e^{iv(e^{i\theta})} = e^{i\theta}$$

where $u(z) = \operatorname{Re} f(z)$, $v(z) = \operatorname{Im} f(z)$. Thus,

$$u(e^{i\theta}) = \log |e^{i\theta}| = 0, \qquad v(e^{i\theta}) = \theta + 2k_\theta \pi$$

for some $k_\theta \in \mathbb{Z}$.

Since f is analytic (and thus continuous) in D, the composition

$$\phi : [0, 2\pi] \xrightarrow{\theta \mapsto e^{i\theta}} S^1 \xrightarrow{v} \mathbb{R} : \theta \mapsto v(e^{i\theta}) = \theta + 2k_\theta \pi$$

is continuous, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is the unit disc. Thus, the map

$$\phi - \mathrm{id} : [0, 2\pi] \to \mathbb{R} : \theta \mapsto k_\theta$$

is a continuous function with image in $\mathbb{Z}$. Since $[0, 2\pi]$ is connected, the image $(\phi - \mathrm{id})([0, 2\pi])$ is a connected subset in $\mathbb{Z}$. So the set $(\phi - \mathrm{id})([0, 2\pi])$ contains a single point $k_0 \in \mathbb{Z}$. But this implies

$$\begin{aligned} v(1) &= v(e^{i0}) = 0 + 2k_0 \pi \\ &= v(e^{i2\pi}) = 2\pi + 2k_0 \pi \end{aligned}$$

which is a contradiction. $\square$