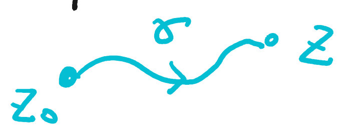
Complex Analysis 4/21

Recall

- (Homotopy Thm) If γ_0, γ_1 are two homotopic closed piecewise C^1 curves in a region D, then
$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$$
for any analytic $f: D \to \mathbb{C}$

- (Closed Curve Thm) Suppose f is analytic in a *simply connected domain D, and C is a closed piecewise C^1 curve in D.
Then $$\int_C f(z)\,dz = 0$$

- (Integral Thm) Suppose f is analytic in a *simply connected domain D.
Let $z_0 \in D$, and
$$F(z) = \int_{z_0}^{z} f(w)\,dw \quad , \qquad \text{for } z \in D$$
where the integration is over a piecewise C^1 curve γ from z_0 to z



Then
(i) $\int_{z_0}^{z} f(w)\,dw$ is independent of choice of γ
(ii) $F'(z) = \dfrac{d}{dz}\left(\int_{z_0}^{z} f(w)\,dw\right) = f(z)$
$\forall z \in D$

Example (p.113)

Suppose f is analytic in the annulus

$$A = \{ z \in \mathbb{C} : 1 < |z| < 4 \}$$

NOT simply connected

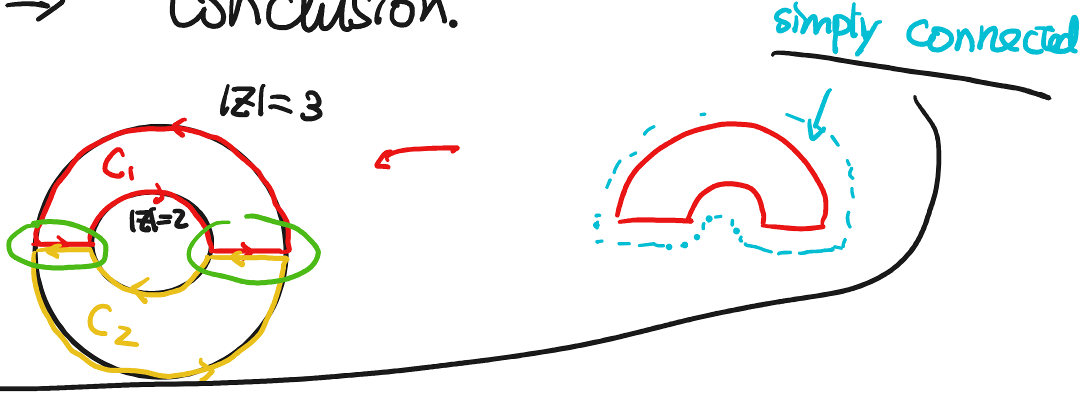
Then

$$\int_{|z|=2} f(z)\,dz = \int_{|z|=3} f(z)\,dz$$

pf

method I Construct a homotopy between the 2 circles $\Rightarrow$ conclusion.

method II



$$\Rightarrow \int_{C_1} f(z)\,dz = 0 \quad \text{(Closed Curve Thm, Thm 8.6)}$$

Similarly,

$$\int_{C_2} f(z)\,dz = 0$$

$$\Rightarrow \quad 0 = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

$$= \int_{|z|=3} f(z)\,dz - \int_{|z|=2} f(z)\,dz \qquad \#$$

Log

Def 8.7

Def.

We say f is an analytic branch of $\log z$ in a domain D if

(1) f is analytic in D

(2) $\exp(f(z)) = z \quad \forall z \in D$

Remark

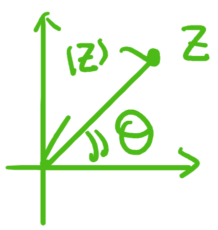
Let f be an analytic branch of $\log z$.

$$e^{f(z)+2\pi k i} = e^{f(z)} \cdot \overbrace{\boxed{e^{2\pi k i}}}^{1}$$

① $g(z) = f(z) + 2\pi k i$ is also an analytic branch of $\log z$ for any fixed $k \in \mathbb{Z}$

② Suppose $u(z) = \mathrm{Re}(f(z))$, $v(z) = \mathrm{Im}(f(z))$

$$\Rightarrow \quad z = \exp(f(z)) = \exp(u(z) + i v(z))$$



$$= e^{u(z)}(\cos v(z) + i \sin v(z))$$

$$= |z|(\cos\theta + i \sin\theta)$$

$$\Rightarrow \quad e^{u(z)} = |z| \Rightarrow u = \log|z|$$

$$v(z) = \mathrm{Arg}\, z = \theta + 2k\pi$$

So
$$\boxed{f(z) = \log|z| + i\,\mathrm{Arg}\, z}$$

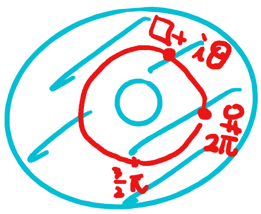
defined up to $2k\pi$

NOT a well-defined function

Remark

$\nexists$ analytic branch of $\log z$ in

$$A = \{ z \in \mathbb{C} : 1 < |z| < 4 \}$$



Thm 8.8

Suppose D is a simply connected domain and $0 \notin D$

Choose $z_0 \in D$, fix a value of $\log z_0$ and set

$$f(z) = \int_{z_0}^{z} \frac{1}{w}\, dw + \log z_0$$

Then $f(z)$ is an analytic branch of $\log z$ in D

pf

① f is well defined and analytic because $\frac{1}{w}$ is analytic in $D \not\ni 0$ and D is simply connected (Integral Thm)

② Let

$$g(z) = z e^{-f(z)} \equiv 1 \qquad -\frac{1}{z}$$

$$\Rightarrow g'(z) = e^{-f(z)} + z \cdot e^{-f(z)} (-f'(z))$$

$$= e^{-f(z)} - e^{-f(z)}$$

$$= 0 \qquad \forall z \in D$$

$$\Rightarrow g(z) = \text{constant} = g(z_0)$$

$$= z_0 e^{-f(z_0)} = z_0 e^{-\log z_0} = \frac{z_0}{z_0} = 1$$

$$\Rightarrow e^{f(z)} = z \quad \forall z \in D \quad \#$$

<u>Remark</u>

$$\exp\left(\tfrac{1}{2}\log z\right)^2 = \exp(\log z) = z$$

So " $\sqrt{z} = \exp\left(\tfrac{1}{2}\log z\right)$ "

More precisely, if f is an analytic branch of $\log z$, then

$$g(z) = \exp\left(\tfrac{1}{2} f(z)\right)$$

is an analytic function s.t.

$$(g(z))^2 = z$$

Similarly,

$$\left(\exp\left(\frac{1}{n} f(z)\right)\right)^n = z.$$

Note

Similar as $\log z$, $\sqrt{z}$ CANNOT be defined on an arbitrary domain $D \subseteq \mathbb{C}$.

Ch9 Isolated singularities of an analytic function

Classification of isolated singularities

Def 9.1 & 9.2

A deleted neighborhood of $z_0 \in \mathbb{C}$ is a set of the form

$$D'(z_0;\varepsilon) = D(z_0;\varepsilon) \setminus \{z_0\}$$

$$= \{z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon\}$$

A function f is said to have an isolated singularity at z_0 if f is analytic in a deleted nbd of z_0 but is NOT analytic at z_0

Suppose f has an isolated singularity at z_0

(i) If $\exists$ function g s.t.

(a) g is analytic at z_0

(b) $f \equiv g$ in some deleted nbd of z_0

we say f has a removable singularity at z_0

(ii) If $\exists$ A, B s.t.

(a) A, B are analytic at z_0

(b) $A(z_0) \neq 0, \quad B(z_0) = 0$

(c) $f \equiv \frac{A}{B}$ in some deleted nbd of z_0.

then we say f has a pole at z_0

If B has a zero of order k at z_0

(i.e. $B(z_0) = B'(z_0) = \cdots = B^{(k-1)}(z_0) = 0, \quad B^{(k)}(z_0) \neq 0$)

then we say f has a pole of order k at z_0

(iii) If f has neither a removable singularity nor a pole at z_0, then we say f has an essential singularity.

Example

(i) $f(z) := \begin{cases} \sin z & , z \neq 2 \\ 0 & , z = 2 \end{cases}$ has a removable singularity at $z=2$

(ii) $g(z) = \frac{1}{z-3}$ has a pole of order 1 at $z=3$

$\frac{1}{(z-3)^5}$ has a pole of order 5 at $z=3$

(iii) $e^{\frac{1}{z}}$ has an essential singularity at $z=0$.

Remark

$\varepsilon > 0$

By Thm 7.7, if f is analytic in $D'(z_0;\varepsilon)$ and continuous at z_0, then f is analytic in $D(z_0;\varepsilon)$.

So if f has an isolated singularity at z_0, then f must be discontinuous at z_0.