Complex Analysis 4/18

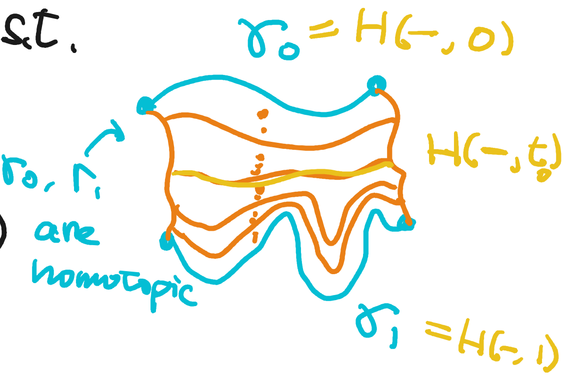
Ch 8 Simply connected domain

Simply connectedness

Def (homotopy)

Let $\gamma_0, \gamma_1 : [a,b] \longrightarrow D$ be two piecewise C^1 curves in $D \subseteq \mathbb{C}$.

A homotopy between piecewise C^1 curves γ_0 and γ_1 is a continuous map $H: [a,b] \times [0,1] \longrightarrow D$ s.t.



① $\forall s_0 \in [a,b]$, the map

$[0,1] \longrightarrow D : t \mapsto H(s_0, t)$

is a piecewise C^1 curve

② $\forall t_0 \in [0,1]$, the map

$[a,b] \longrightarrow D : s \mapsto H(s, t_0)$

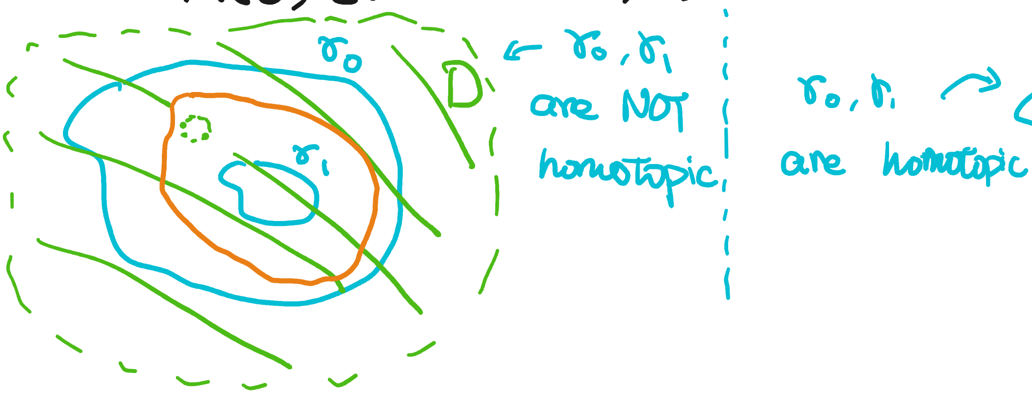
is a piecewise C^1 curve.

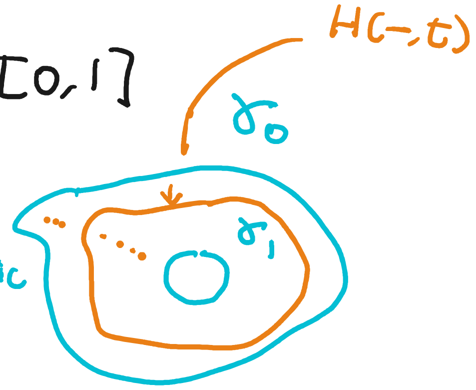
③ $H(s,0) = \gamma_0(s)$, $H(s,1) = \gamma_1(s)$ $\forall s \in [a,b]$

We say γ_0 is homotopic to γ_1 if $\exists$ such a homotopy H.

If γ_0, γ_1 are closed (i.e. $\gamma_j(0) = \gamma_j(1)$),
we require a homotopy H satisfies

④ $H(0,t) = H(1,t) \quad \forall t \in [0,1]$





<u>Def</u> (cf. <u>Def 8.1</u>)

An open connected set $D \subseteq \mathbb{C}$ is called <u>simply connected</u> if every closed piecewise C^1 curve in D is homotopic to a constant curve

<u>Remark</u>

① "simply connected" can be defined for any topological spaces

$\pi_0 = \pi_1 = 0$ i.e. connected + $\pi_1 = 0$ (fundamental group)

<u>Def 8.1</u> in textbook only works for domains in $\mathbb{C}$

② Graphically,

"simply connected" = connected + "no holes" (2-dim) simply connected ... connected ...

$D(\alpha;r)$

Example

① For $r \in (0,\infty]$, $\alpha \in \mathbb{C}$, $D(\alpha;r)$ is simply connected

pf. $\forall$ closed piecewise C^1 curve $\gamma : [a,b] \to D(\alpha;r)$,

$H(s,t) = (1-t)\gamma(s) + t\cdot\alpha$

is a homotopy between γ and $\gamma_1 \equiv \alpha$. $\blacksquare$

② $D(0;1) - \{0\}$ is NOT simply connected.

③ $D(0;1) \cup D(3;1)$ is NOT simply connected because it's NOT connected.

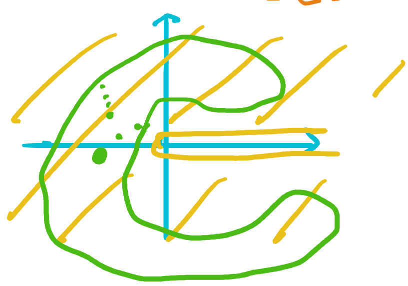
④ The annulus

$$A = \{z \in \mathbb{C} : 1 < |z| < 3\}$$

is NOT simply connected

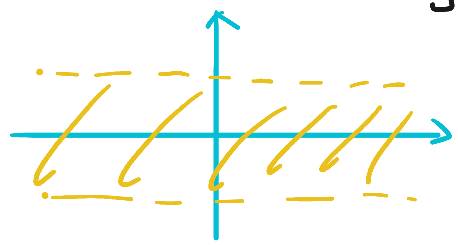
⑤ $\mathbb{C} - \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0,\ \mathrm{Im}(z) = 0\}$

is simply connected



⑥ The strip $S = \{z \in \mathbb{C} : -1 < \mathrm{Im}\, z < 1\}$

is simply connected



exer

"star-like" $\Rightarrow$ simply connected

General closed curve thm

Recall (Closed curve thm, Thm 6.3)

Suppose $r \in (0, \infty]$. If $f: D(z_0; r) \to \mathbb{C}$ is analytic, then $\forall$ closed piecewise C^1 curve C in $D(z_0; r)$, we have

$$\int_C f(z)\,dz = 0$$

This section: this can be replaced by a simply connected domain

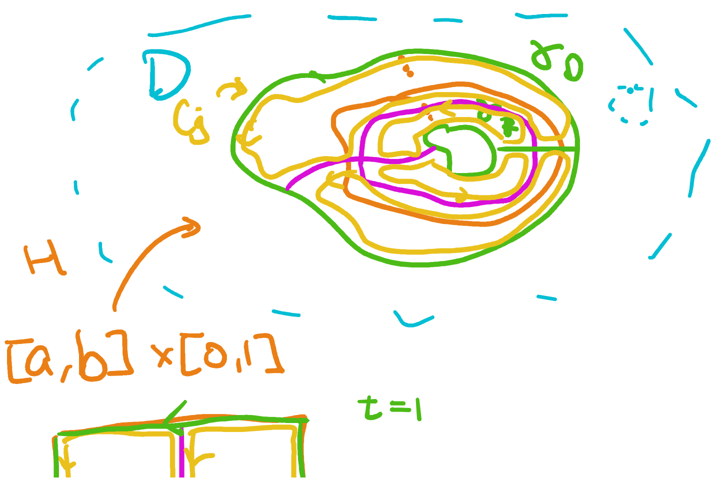
Thm (Homotopy Thm)

If γ_0 and γ_1 are two homotopic closed piecewise C^1 curve in a region D, then

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$$

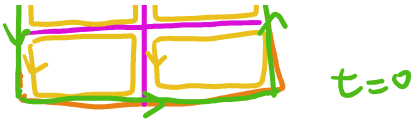
for any analytic function $f: D \to \mathbb{C}$.

idea of pf



Divide $[a,b]\times[0,1]$ so that the image of each piece under H is in an open disc $D(z_0; \varepsilon)$

By Thm 6.3.



$$\int_{C_j} f(z)\,dz = 0 \quad \forall j$$

$$\Rightarrow \int_{\gamma_0} f(z)\,dz - \int_{\gamma_1} f(z)\,dz$$

$$= \sum_j \int_{C_j} f(z)\,dz = 0 \qquad \#$$

<u>Cor</u> (<u>Thm 8.5</u>, General Integral Thm)

Let D be a simply connected domain and $f: D \to \mathbb{C}$ be analytic. Then there exists an analytic function $F: D \to \mathbb{C}$ s.t.

$$F'(z) = f(z) \qquad \forall z \in D$$

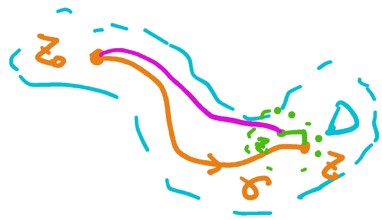
<u>pf</u>

Fix $z_0 \in D$. Since D is open connected,

from Advanced Calculus

$\forall z \in D$, $\exists$ a piecewise C^1 curve $\gamma: [0,1] \to D$ s.t.

$$\gamma(0) = z_0, \quad \gamma(1) = z$$



Define

$$F(z) := \int_\gamma f(w)\,dw$$

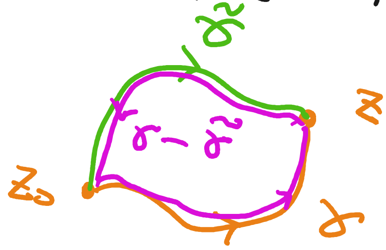
① F is well-defined:

If $\tilde{\gamma}$ is another piecewise C^1 curve s.t.

$$\tilde{\gamma}(0) = z_0, \quad \tilde{\gamma}(1) = z$$

then the close curve

$$\gamma - \tilde{\gamma}$$



is homotopic to a constant curve γ_0 (because D is simply connected)

$$\Rightarrow \int_{\gamma-\tilde{\gamma}} f(z)\,dz = \int_\gamma f(z)\,dz - \int_{\tilde{\gamma}} f(z)\,dz$$

Homotopy Thm $\|$

$$\int_{\gamma_0} f(z)\,dz = \int_a^b f(\gamma_0(t)) \cdot \underset{\overset{\shortparallel}{0}}{\gamma_0'(t)}\,dt = 0$$

So $F(z)$ is independent of the choice of γ.

② Compare $F(z)$ and the construction in the proof of Integral Thm (Thm 6.2). (call this one $\tilde{F}$)

One can show

$$F(z) = \tilde{F}(z) + \text{a constant}$$

$$\Rightarrow \quad F'(z) = \tilde{F}'(z) = f(z) \qquad \#$$

Cor (Thm 8.6, General Closed Curve Thm)

Suppose f is analytic in a simply connected domain D and C is a closed piecewise C^1 curve in D. Then

$$\int_C f(z)\, dz = 0$$

<u>pf</u>

Since D is simply connected, C is homotopic to a constant curve γ_0

$$\overset{\text{Homotopy}}{\underset{\text{Thm}}{\Longrightarrow}} \quad \int_C f(z)\,dz = \int_{\gamma_0} f(z)\,dz = 0 \qquad \#$$