Complex Analysis 4/14

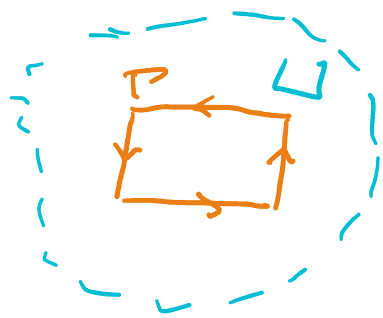
Converse of rectangle thm

Recall (Rectangle Thm i.e. Thm 6.1)

Suppose f is analytic in $U \underset{\text{open}}{\subseteq} \mathbb{C}$ and $R \subseteq U$ is a rectangle. Then

$$\int_{\Gamma} f(z)\,dz = 0$$

where $\Gamma = \partial R$



Morera Thm (Thm 7.4)

Let f be a continuous function on an open set $U \subseteq \mathbb{C}$. If

$$\int_{\Gamma} f(z)\,dz = 0$$

whenever Γ is the boundary of a closed rectangle in U, then f is analytic in U.

pf (same as the proof of Integral Thm i.e. Thm 4.5)

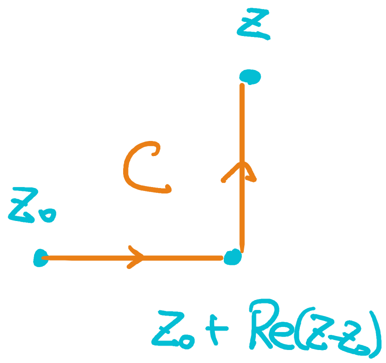
Let $z_0 \in U$, $\varepsilon > 0$ s.t. $D(z_0;\varepsilon) \subseteq U$.

Let

$$F(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta$$

$$= \int_C f(\zeta)\, d\zeta$$

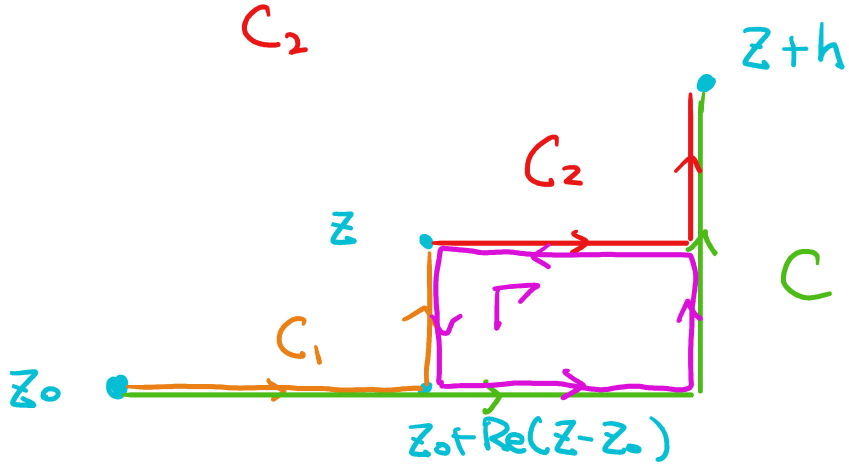
where $z \in D(z_0; \varepsilon)$, and C is



By assumption,

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\left(\underset{C}{\int_{z_0}^{z+h}} f(\zeta) d\zeta - \underset{C_1}{\int_{z_0}^{z}} f(\zeta) d\zeta \right)$$

$$= \frac{1}{h}\left(\underset{C_2}{\int_{z}^{z+h}} f(\zeta) d\zeta + \underbrace{\int_\Gamma f(\zeta) d\zeta}_{\overset{\text{by assumption}}{=} 0} \right)$$



$$\Rightarrow \left| \frac{F(z+h) - F(z)}{h} - f(z) \right| = \frac{1}{|h|} \left| \int_z^{z+h} f(\zeta) - f(z)\, d\zeta \right|$$

$$\leq \frac{1}{|h|} \overset{\leq 2|h|}{\underline{\text{length}(C_2)}} \cdot \left(\sup_{\zeta \in C_2} | f(\zeta) - f(z) | \right)$$

$$\leq 2 \left(\sup | f(\zeta) - f(z) | \right) \longrightarrow 0 \quad \text{as } h \to 0$$

$\zeta \in G$ by continuity of f

So F is analytic in $D(z_0; \varepsilon)$ and

$$F'(z) = f(z) \quad \text{in } D(z_0; \varepsilon)$$

$\Rightarrow$ Since z_0 is arbitrary in U, we conclude f is analytic in U #

Def 7.5

Suppose $\{f_n\}$ and f are defined in D. We say f_n converges to f uniformly on compacta if $f_n \to f$ uniformly on every compact subset $K \subseteq D$.

Thm 7.6

Suppose f_n are analytic in an open domain D and $f_n \to f$ uniformly on compacta. Then f is analytic in D.

pf

Given any $z_0 \in D$, $\exists \varepsilon > 0$ s.t. $\overline{D(z_0; \varepsilon)} \subseteq D$

Since $\overline{D(z_0; \varepsilon)}$ is compact, $f_n \to f$ uniformly in $U = D(z_0; \varepsilon)$

Since f_n are continuous, f is also continuous in U

Let Γ be the boundary of any rectangle in U.

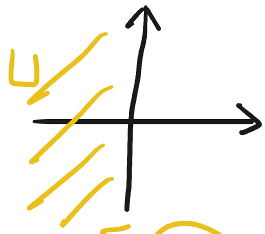
$$\int_\Gamma f(z)\,dz = \int_{\Gamma \subseteq U} \lim_{n\to\infty} f_n(z)\,dz$$

$$\overset{f_n \to f}{\underset{\text{unif on } \Gamma \subseteq U}{=}} \lim_{n\to\infty} \underbrace{\int_\Gamma f_n(z)\,dz}_{=0} = 0$$

By Morera Thm, f is analytic in U (z_0 arbitrary in D)

$\Rightarrow$ f is analytic in D. $\#$

<u>Example</u> (p. 98–99)



The function

$$f(z) := \int_0^\infty \frac{e^{zt}}{t+1}\,dt \qquad (t \in \mathbb{R})$$

is analytic in the left half plane

$$U = \{ z \in \mathbb{C} : \operatorname{Re} z < 0 \}$$

pf

Let $$f_n(z) = \int_0^n \frac{e^{zt}}{t+1}\,dt$$

<u>Step 1</u>

Let Γ be the boundary of an arbitrary rectangle in $\sqcup$.

By Fubini Thm,

$$\int_\Gamma f_n(z)\,dz = \int_\Gamma \int_0^n \frac{e^{zt}}{t+1}\,dt\,dz$$

entire $\forall\, t \geq 0$

$$= \int_0^n \underbrace{\int_\Gamma \frac{e^{zt}}{t+1}\,dz}_{=0}\,dt$$

$$= \int_0^n 0\,dt = 0$$

$$\int_a^b \int_0^n \frac{e^{t\cdot z(s)}}{t+1}\,dt\; z'(s)\,ds$$

$z(s)$, $a \leq s \leq b$ is a parametriation of Γ

$$= \int_a^b \int_0^n \operatorname{Re}\left(\frac{e^{t\cdot z(s)}}{t+1}\cdot z'(s)\right) dt\,ds$$

$$+ i \int_a^b \int_0^n \operatorname{Im}(\text{---}\cdots)\,dt\,ds$$

Then apply Fubini in Calculus

By Morera Thm, f_n are analytic in $\sqcup$ $\forall n$.

Step 2

$$|f_n(z) - f(z)| = \left|\int_n^\infty \frac{e^{zt}}{t+1}\,dt\right|$$

$$e^{t\operatorname{Re}(z) + i t \operatorname{Im}(z)}$$

$$= e^{t\operatorname{Re}(z)}(\cos + i \sin)$$

$$\leq \int_n^\infty \left|\frac{e^{zt}}{t+1}\right| dt$$

$$|\cdot| = e^{t\operatorname{Re}(z)}$$

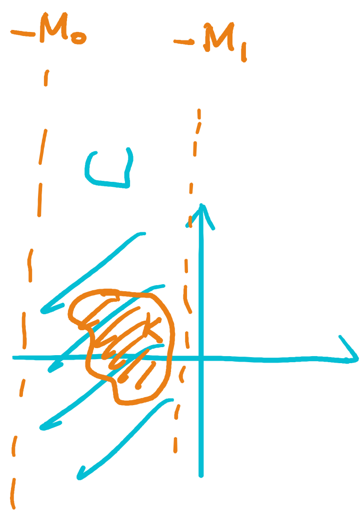
$$\leq \int_n^\infty |e^{zt}|\,dt$$

$\int^\infty \quad t\, \operatorname{Re}(z)$

$$= \int_n^{\infty} e^{t\,\mathrm{Re}(z)}\, dt$$

$$= \frac{1}{\mathrm{Re}(z)} e^{t\,\underbrace{\mathrm{Re}(z)}_{}}\Big|_{t=n}^{\infty} \qquad \mathrm{Re}(z) < 0$$

$$= \frac{-1}{\mathrm{Re}(z)} e^{n\,\mathrm{Re}(z)}$$



For any compact $K \subseteq U$, $\exists\ M_0, M_1 > 0$ ← No need

s.t.

$$-M_0 \le \mathrm{Re}(z) \le -M_1 \qquad \forall z \in K$$

$\Rightarrow$

$$|f_n(z) - f(z)| \le \frac{-1}{\mathrm{Re}(z)} e^{n\,\mathrm{Re}(z)}$$

$$\le \frac{1}{M_1}\left(e^{M_1}\right)^{-n} \longrightarrow 0 \quad \text{as } n \to \infty$$

↖ indep of $z \in K$

So $f_n \to f$ uniformly in K.

By Thm 7.6, f is analytic in U #

Thm 7.7

Suppose f is continuous in an open set D and analytic in D except possibly at the points of a line segment.

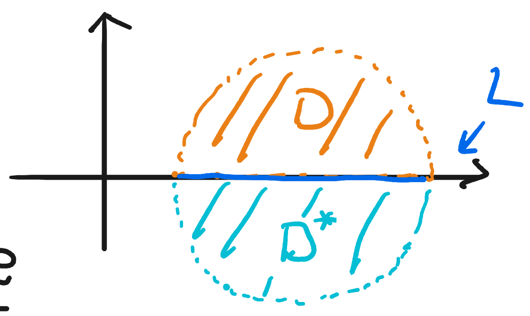
Then f is analytic throughtout D.

pf: Apply Morera Thm. Skip.

Reflection principle

Let D be a region which contained in the upper or lower half plane

Suppose ∂D contains a segment L on the real axis



Schwarz Reflection Principle

(Thm 7.8)

Suppose f is continuous in $\bar{D}$ and ananalytic in D (called "C-analytic" in book) and $f(L) \subseteq \mathbb{R}$.

Then the function

$$g(z) := \begin{cases} f(z), & z \in D \cup L \\ \overline{f(\bar{z})} & z \in D^* \end{cases}$$

is analytic in $D \cup L \cup D^*$, where

$$D^* = \{ z \in \mathbb{C} : \bar{z} \in D \}$$

pf

① Since $g \equiv f$ in D, g is analytic in D

② If $z \in D^*$, $\forall h$ s.t. $z+h \in D^*$, we have

$$\frac{g(z+h) - g(z)}{h} = \frac{\overline{f(\bar{z}+\bar{h})} - \overline{f(\bar{z})}}{h}$$

$$= \overline{\left(\frac{f(\bar{z}+\bar{h}) - f(\bar{z})}{\bar{h}} \right)}$$

$$\longrightarrow \overline{f'(\bar{z})} \quad \text{as } h \to 0$$

$\Rightarrow$ g is analytic in D^*

③ Since f is continous and real on L, g is also continuous on L.

So, by Thm 7.7, g is analytic throughout $D \cup L \cup D^*$. #

Cor 7.9

" $D \cup L \cup D^*$ "

If f is analytic in a region symmetric with respect to the real axis and if f is real for real z, then

$$f(z) = \overline{f(\bar{z})}$$

pf: Thm 7.8 + Uniqueness Thm. $\square$