Complex Analysis 4/7

<u>Recall</u>

- <u>Maximum Modulus Thm</u> (Thm 6.13)
  Suppose f is a nonconstant analytic function in a region D. Then $|f|$ cannot assume a maximum at a interior point.

- <u>Minimum Modulus Thm</u> (Thm 6.14)
  If f is a nonconstant analytic function in a region D, and $\exists\ \delta > 0$ s.t.
  $$|f(z)| \leq |f(\omega)| \quad \forall\ \omega \in D(z;\delta) \cap D,$$
  then $|f(z)| = 0$. (e.g. $f(z) = z$, $|f|$ has min at $z=0$)

- A map (between topological / metric spaces) is called <u>open</u> if it maps open sets to open sets.

<u>Open Mapping Thm</u> (Thm 7.1)
A nonconstant analytic map is open.

pf

Let $g : U \xrightarrow[\text{open}]{\subset \mathbb{C}} \mathbb{C}$ be nonconstant and analytic. Let $\alpha \in U$, and

$$f(z) := g(z) - g(\alpha)$$ (Want to show: $g(U)$ is open)

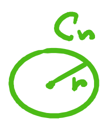
$\Rightarrow$ $f(\alpha) = 0$

Claim

$\exists\ C_r = \{ \alpha + re^{i\theta} : \theta \in [0, 2\pi] \}$ s.t.

$\overline{D(\alpha; r)} \subseteq U$ and $f(z) \neq 0 \quad \forall z \in C_r$

pf



If not, $\exists N, \exists \theta_n \in [0, 2\pi]$ s.t.

$$f(\alpha + \tfrac{1}{n} e^{i\theta_n}) = 0 \quad \forall n \geq N$$

$\Rightarrow$ $f = 0$ in $\{ \alpha + \frac{1}{n} e^{i\theta_n} : n \geq N \}$ has an acc pt $\alpha \in U$

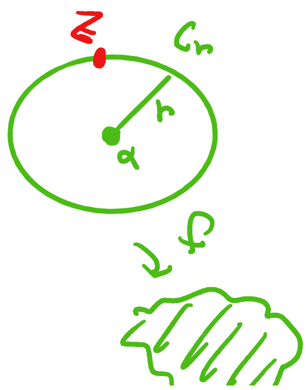
$\overset{\text{uniqueness}}{\underset{\text{thm}}{\Rightarrow}}$ $f \equiv 0$ in U $\rightarrow\leftarrow$ to noncontant #

Let C be such a C_r, and

$$\varepsilon = \frac{1}{2} \min_{z \in C} |f(z)| > 0$$

Claim

$D(0; \varepsilon) \subseteq f(D(\alpha; r))$



pf

Let $\omega \in D(0; \varepsilon)$. For $z \in C_r$ ($= C$),

$$|f(z) - \omega| \geq |f(z)| - |\omega| \geq 2\varepsilon - \varepsilon = \varepsilon$$

and at α,

$$|f(\alpha) - \omega| = |\omega| < \varepsilon \leq |f(z) - \omega|$$

$\Rightarrow$ $|f(z) - \omega|$ assumes its minimum inside $C_r = C$

Assume $z_0 \in D(\alpha; r)$ s.t.

$$|f(z_0) - \omega| = \min_{z \in \overline{D(\alpha; r)}} |f(z) - \omega|$$

By <u>Minimum Modulus Thm</u>,

$$f(z_0) - \omega = 0$$

$\Rightarrow$ $\underset{\substack{\Downarrow \\ D(0;\varepsilon)}}{\omega} \in f(D(\alpha; r))$

$\Rightarrow$ $D(0; \varepsilon) \subseteq f(D(\alpha; r))$ #

recall: $f(z) = g(z) - g(\alpha)$

So

$$D(g(\alpha); \underset{\varepsilon}{\overset{\varepsilon_\alpha \, \shortparallel}{\vphantom{\varepsilon}}}) = D(0; \varepsilon) + g(\alpha) \subseteq f(D(\alpha; r)) + g(\alpha)$$

$$= g(D(\alpha; r)) \subseteq g(U)$$

$\Rightarrow$ $g(U) = \bigcup_{\alpha \in U} D(g(\alpha); \varepsilon_\alpha)$ is open #

<u>Functions on unit disc</u>

$D(0;1) = \{ z \in \mathbb{C} : |z| < 1 \}$

<u>Schwarz Lemma</u> (Thm 7.2)

Suppose that (a) f is analytic in the unit disc $D = D(0;1)$, (b) $|f(z)| \leq 1$ $\forall z \in D$ and (c) $f(0) = 0$.

Then

(i) $|f(z)| \leq |z|$ $\forall z \in D$

(ii) $|f'(0)| \leq 1$

with equality holds in (i) or (ii) at some point iff $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$.

pf

Let

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & 0 < |z| < 1 \\ f'(0) & z = 0 \end{cases}$$

which is analytic by Prop 6.7.

By (b), $|g(z)| \leq \dfrac{1}{|z|} \longrightarrow 1$ as $|z| \to 1$

By Maximum Modulus Thm,

$$|g(z)| \leq 1 \quad \forall z \in D$$

$$\Rightarrow \begin{cases} |f(z)| \leq |z| \\ |f'(0)| \leq 1 \end{cases}$$

"=" holds at some point $z_0 \in D$

$\Leftrightarrow$ $|g(z_0)| = 1$ for some $z_0 \in D$

$\overset{\text{Max Mod}}{\Leftrightarrow}$ $g = \text{constant} = g(z_0) = e^{i\theta}$ for some inm

$\theta \in \mathbb{R}$

$\Leftrightarrow \quad f(z) = e^{i\theta} z$ #

Prop (p. 95)

Suppose $|\alpha| < 1$ and

$$B_\alpha(z) := \frac{z-\alpha}{1-\bar{\alpha} z}.$$

Note
- $B_\alpha(\alpha) = 0$
- $B_0(z) = z$

Then $B_\alpha : D = D(0;1) \longrightarrow D_{\subseteq \mathbb{C}}$ is analytic.

pf

Note that $1-\bar{\alpha} z = 0 \Leftrightarrow z = \dfrac{1}{\bar{\alpha}} \notin D$

And $\left|\dfrac{1}{\bar{\alpha}}\right| = \dfrac{1}{|\alpha|} > 1$

$\Rightarrow$ B_α is analytic in D

If $|z| = 1$, then

$$|B_\alpha(z)|^2 = B_\alpha(z)\overline{B_\alpha(z)}$$

$$= \frac{z-\alpha}{1-\bar{\alpha} z} \cdot \frac{\bar{z}-\bar{\alpha}}{1-\alpha\bar{z}}$$

$$= \frac{\overset{z\bar{z}}{1} + |\alpha|^2 - (z\bar{\alpha} + \bar{z}\alpha)}{1+|\alpha|^2 - (\bar{\alpha} z + \alpha\bar{z})} = 1$$

By Maximum Modulus Thm,

i.e. $B_\alpha : D \to D$

$$|B_\alpha(z)| < 1 \qquad \forall\ |z| < 1$$ #

Example (p.95-96)

① Suppose f is analytic and bounded by 1 in the unit disc. and $f(\frac{1}{2})=0$

Show that

$$|f(\tfrac{3}{4})| \leq \frac{2}{5}$$

and "=" is achieved by some analytic function

pf

Let

$|f(z)|\leq 1$ $\quad$ $|B_{\frac{1}{2}}(z)|=1$ when $|z|=1$

$$g(z) := \begin{cases} f(z) \big/ B_{\frac{1}{2}}(z) = f(z) \Big/ \left(\dfrac{z-\frac{1}{2}}{1-\frac{1}{2}\cdot z}\right) & z \neq \frac{1}{2} \\ \frac{3}{4} f'(\frac{1}{2}) = \lim\limits_{z\to\frac{1}{2}} \dfrac{f(z)}{B_{\frac{1}{2}}(z)} & z = \frac{1}{2} \end{cases}$$

$(1-\frac{1}{2}z)\big|_{z=\frac{1}{2}} = \frac{3}{4}$

$\Rightarrow$ $g(z)$ is analytic and

$$\overline{\lim_{|z|\to 1}} |g(z)| \leq 1$$

$|g(z)| < |g(z_0)|$

By Maximum Modulus Thm,

If not, $\exists z_0 \in D$ s.t. $|g(z_0)| > 1$

$\Rightarrow \max_{\text{blue}} |g(z)| > 1$

$$|g(z)| \leq 1 \qquad \forall\ |z|<1$$

$$\Rightarrow \boxed{|f(z)| \leq |B_{\frac{1}{2}}(z)| \qquad \forall\ |z|<1}$$

In particular,

$$|f(\tfrac{3}{4})| \leq |B_{\frac{1}{2}}(\tfrac{3}{4})| = \frac{2}{5}$$

and " = " holds if $f = B_{\frac{1}{2}}$. #

② Suppose $f : D = D(0;1) \to D$ is analytic

$$|f'(\tfrac{1}{3})| = \max\{|h'(\tfrac{1}{3})| : h : D \to D \text{ analytic}\}$$

↖ In fact, this max is assumed by $B_{\frac{1}{3}}(z)$ (exer 10, 11 in Ch 7)

Show that $f(\frac{1}{3}) = 0$.

pf

Suppose $f(\frac{1}{3}) \neq 0$ and consider

$$g(z) = \frac{f(z) - f(\frac{1}{3})}{1 - \overline{f(\frac{1}{3})}\, f(z)} = B_{f(\frac{1}{3})}(f(z))$$

We have

$$g : D \xrightarrow{f} D \xrightarrow{B_{f(\frac{1}{3})}} D$$

$(\Rightarrow |g(z)| < 1 \ \forall\, |z| < 1)$

Note that

$$\left|g'(\tfrac{1}{3})\right| = \left| \frac{f'(\frac{1}{3}) \cdot \cancel{(1 - \overline{f(\frac{1}{3})} f(\frac{1}{3}))} - \underbrace{\cancel{(f(\frac{1}{3}) - f(\frac{1}{3}))}}_{=0} \cdot (\cdots)}{(1 - \overline{f(\frac{1}{3})} f(\frac{1}{3}))^{\cancel{2}}} \right|$$

$$= \left| \frac{f'(\frac{1}{3})}{1 - |f(\frac{1}{3})|^2} \right| > |f'(\tfrac{1}{3})|$$

$$\left| 1 - \underbrace{\left| 1+\left(\tfrac{1}{3}\right) \right|}_{0 \lneq \; < 1} \right|$$

$$\left(\longrightarrow \longleftarrow \right)$$

#