Complex Analysis 3/31

Recall

- If two functions f and g, analytic in a region (open connected set) D,
  $$f = g$$
  on a set with an acc point in D,
  then $$f \equiv g \quad \text{in} \quad D$$

- We were proving

Thm 6.11

If f is entire and $f(z) \to \infty$ as $z \to \infty$, then f is a polynomial

pf

By assumption, $\exists B > 0$ s.t. $|z| > B \Rightarrow |f(z)| > 1$

We proved:

Claim 1

f has finite zeros (we used uniqueness thm here)
$\alpha_1, \alpha_2, \cdots, \alpha_N$

Claim 2

Let k_j be the number

$$f(\alpha_j) = f^{(1)}(\alpha_j) = \cdots = f^{(k_j-1)}(\alpha_j) = 0$$

$$f^{(k_j)}(\alpha_j) \neq 0$$

Then $k_j < \infty$ (i.e. $\exists\, k_j$ s.t. $f^{(k_j)}(\alpha_j) \neq 0$)

(we used Taylor expansion of f here)

Let

$$g(z) = \begin{cases} \dfrac{f(z)}{(z-\alpha_1)^{k_1}(z-\alpha_2)^{k_2}\cdots(z-\alpha_N)^{k_N}} & \text{if } z \neq \alpha_1, \cdots, \alpha_N \\[2ex] \lim\limits_{z\to\alpha_j} \dfrac{f(z)}{(z-\alpha_1)^{k_1}\cdots(z-\alpha_N)^{k_N}} & \text{if } z = \alpha_j \end{cases}$$

By applying Cor 5.9 repeatedly, we know g is entire

Recall (Cor 5.9)

Suppose f is analytic in $U \underset{\text{open}}{\subset} \mathbb{C}$,

distinct $a_1, \cdots, a_N \in U$ and $f(a_k) = 0$, $k = 1, \cdots, N$

Then

$$\tilde{g}(z) := \begin{cases} \dfrac{f(z)}{(z-a_1)(z-a_2)\cdots(z-a_N)} & \forall z \in U \setminus \{a_1, \cdots, a_N\} \\[2ex] \lim\limits_{z\to a_j} \dfrac{f(z)}{(z-a_1)\cdots(z-a_N)} & z = a_j \end{cases}$$

is also analytic in U

Also note that $g(z) \neq 0$ $\forall z \in \mathbb{C}$ (exer.)

$\Rightarrow \underbrace{h(z) := \dfrac{1}{g(z)}}$ $\neq 0 \ \forall z \in \mathbb{C}$ is also entire

($\Rightarrow$ continuous)

Since h is continuous and

$\{|z| \leq B\}$ is compact,

there exists A s.t. $|h(z)| \leq A \quad \forall\, |z| \leq B$

For $|z| > B$, $|f(z)| > 1$

$$\Rightarrow |h(z)| = \frac{|z-\alpha_1|^{k_1} \cdots |z-\alpha_N|^{k_N}}{|f(z)|} < |z-\alpha_1|^{k_1} \cdots |z-\alpha_N|^{k_N}$$

$$\leq (|z|+|\alpha_1|)^{k_1} \cdots (|z|+|\alpha_N|)^{k_N}$$

Note
$|\alpha_j| \leq B < |z|$

$$\leq (|z|+|z|)^{k_1} \cdots (|z|+|z|)^{k_N}$$

$$= 2^{k_1+k_2+\cdots+k_N} \cdot |z|^{k_1+\cdots+k_N}$$

Let $d = k_1 + \cdots + k_N$, $E = 2^{k_1+\cdots+k_N}$. Then

$$|h(z)| \leq A + E \cdot |z|^d \qquad \forall z \in \mathbb{C}$$

So, by Liouville Thm (Thm 5.11),

$h(z)$ is a polynomial.

Since $h(z) \neq 0$ $\forall z \in \mathbb{C}$, by Fundamental Thm of Algebra, h is a constant $C \neq 0$.

$= \frac{1}{g(z)}$

$$\Rightarrow \quad g(z) \equiv \frac{1}{c} = \text{constant}$$

$$\Rightarrow \quad f(z) = \frac{1}{c} \cdot (z-\alpha_1)^{k_1} \cdots (z-\alpha_N)^{k_N}$$

is a polynomial #

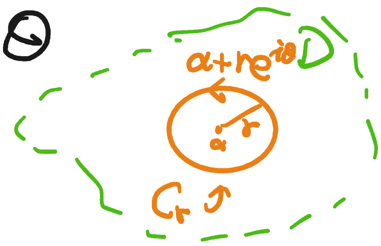
Max / min modulus thm

Mean Value Thm (Thm 6.12)

If f is analytic in D and $\alpha \in D$, then

$$f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha + re^{i\theta})\, d\theta$$

when $\overline{D(\alpha; r)} \subseteq D$



pf

By Cauchy Integral Formula (Thm 6.4),

$$f(\alpha) = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z-\alpha}\, dz$$

where C_r is the circle $\alpha + re^{i\theta}$. $0 \le \theta \le 2\pi$.

$$\Rightarrow \quad f(\alpha) = \frac{1}{2\pi \cancel{i}}\int_0^{2\pi} \frac{f(\alpha+re^{i\theta})}{(\cancel{\alpha}+\cancel{re^{i\theta}}) - \cancel{\alpha}}\, \cancel{i}\cancel{r}\cancel{e^{i\theta}}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\alpha+re^{i\theta})\, d\theta$$

#

Maximum Modulus Thm (Thm 6.13)

Suppose f is a nonconstant analytic function in a region D. Then $\forall z \in D$, $\forall \delta > 0$, $\exists\, \omega \in D(z;\delta) \cap D$ s.t.

$$|f(\omega)| > |f(z)|$$

i.e. $|f|$ cannot have (local) max in D.

pf

Suppose $\exists\, z_0\ \exists\, \delta > 0$ s.t.

$$|f(\omega)| \le |f(z_0)| \quad \forall\, \omega \in D(z_0;\delta) \cap D$$

Let $r > 0$ be any positive number s.t.

$\rightarrow$ $\overline{D(z_0;r)} \subseteq D(z_0;\delta) \cap D$ open in $\mathbb{C}$

$$\Rightarrow \quad f(z_0) \underset{\text{Thm 6.12}}{\overset{\text{by MVT}}{=}} \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$$

$$\Rightarrow \quad |f(z_0)| = \frac{1}{2\pi} \left| \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta \right|$$

Recall (Lemma 4.9)

$\left| \int_a^b G(t)\,dt \right| \le \int_a^b |G(t)|\,dt$

$$\le \frac{1}{2\pi} \int_0^{2\pi} \underbrace{|f(z_0 + re^{i\theta})|}_{\le |f(z_0)|}\, d\theta$$

$$\le \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)|\, d\theta$$

$$= |f(z_0)|$$

$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\,d\theta$

$\shortparallel$

$$\Rightarrow \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{i\theta})|\,d\theta = |f(z_0)| \\ |f(z_0+re^{i\theta})| \leq |f(z_0)| \quad \forall\, \theta\in[0,2\pi] \end{cases}$$

$$\Rightarrow \quad \frac{1}{2\pi}\int_0^{2\pi} \underbrace{|f(z_0)| - |f(z_0+re^{i\theta})|}_{\geq 0\ ,\ \text{continuous}}\,d\theta = 0$$

$$\Rightarrow \quad |f(z_0)| = |f(z_0+re^{i\theta})| \quad \forall\, \theta\in[0,2\pi]$$

Thus,

$$|f(\omega)| = |f(z_0)| \qquad \forall\, \omega \in D(z_0;r)$$

<u>Recall</u> (<u>Prop 3.7</u>)

If g is analytic in a region U and $|g|$ is constant in U, then g is constant

So f is constant in $D(z_0;r)$

$\xRightarrow[\text{thm}]{\text{uniqueness}}$ f is constant in D ($\longleftarrow\!\!\!\!\times\!\!\!\longrightarrow$) #

<u>Cor</u>

Suppose f is analytic in a bounded

Suppose f is analytic ...

region D and continuous on $\overline{D}$, then $|f|$ has a maximum in the boundary ∂D of D.

pf

Since $|f|$ is continuous on $\overline{D}$ and $\overline{D}$ is compact, $|f|$ has a maximum at a point $p \in \overline{D}$.

By <u>Max Modulus Thm</u>, $p \notin D$

$\Rightarrow p \in \overline{D} \setminus D = \partial D$ $\#$

<u>Minimum Modulus Thm</u> (Thm 6.14)

If f is a nonconstant analytic function in a region D, and $\exists\ \delta > 0$ s.t.

$$|f(z)| \leq |f(\omega)| \quad \forall\ \omega \in D(z;\delta) \cap D$$

then $f(z) = 0$

pf

Suppose $f(z) \neq 0$.

Then $g = \dfrac{1}{f}$ is analytic in $D(z;\delta) \cap D$

and

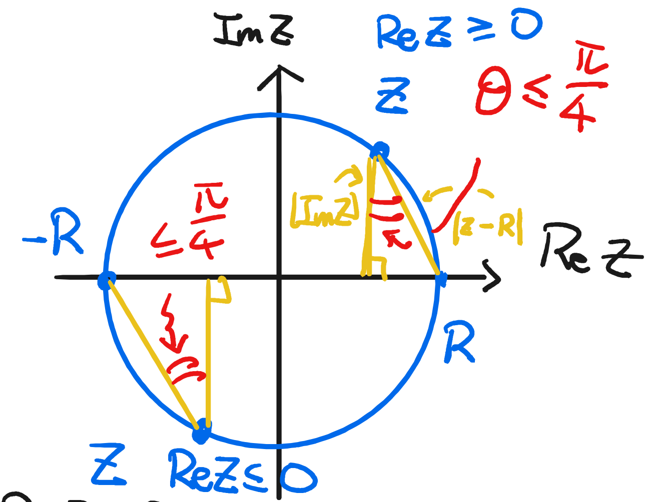
and
$$|g(z)| = \frac{1}{|f(z)|} \geq \frac{1}{|f(\omega)|} = |g(\omega)|$$
$\forall\ \omega \in D(z;\delta) \cap D$

Max Modulus Thm
$\Rightarrow$ $\qquad g = \text{constant}\ (\rightarrow\!\!\!\!\times\!\!\leftarrow)$ #

Application : a Liouville-type theorem

Prop 7.3

If f is an entire function satisfying
$$|f(z)| \leq \frac{1}{|\operatorname{Im} z|} \qquad \forall z \in \mathbb{C}$$
then $\quad f \equiv 0$



pf

Let
$$g(z) = (z^2 - R^2) f(z)$$

Given any z s.t. $|z| = R$, $\operatorname{Re} z \geq 0$
$$|(z-R) f(z)| \leq |z-R| / |\operatorname{Im} z| = \sec\theta$$
for some $\theta \in [0, \frac{\pi}{4}]$

$$\Rightarrow \quad |(z-R) f(z)| \leq \sqrt{2}$$

Similar, if $|z| = R$, $\operatorname{Re} z \leq 0$, then

$$|(z+R)f(z)| \le \sqrt{2}$$

Thus, for any z, $|z|=R$,

$$|g(z)| = |z+R|\,|z-R|\,|f(z)|$$

$$\le \begin{cases} |z+R|\sqrt{2} \le (|z|+R)\sqrt{2} = 2\sqrt{2}R \le 3R & \text{if } \operatorname{Re} z \ge 0 \\ |z-R|\cdot\sqrt{2} \le 2\sqrt{2}R \le 3R & \text{if } \operatorname{Re} z \le 0 \end{cases}$$

$$\le 3R \qquad \forall\ |z| = R$$

By <u>Maximum Modulus Thm</u>,

$$\underset{\displaystyle |z^2-R^2|\,|f(z)|}{\underset{\shortparallel}{|g(z)|}} \le 3R \qquad \forall\ |z| \le R$$

for any fixed z
any $R > |z|$

$$\Rightarrow \quad |f(z)| \le \frac{3R}{|z^2-R^2|} \qquad \forall\ R > |z|$$

$$\Rightarrow \quad |f(z)| = \varlimsup_{R\to\infty} |f(z)| \le \varlimsup_{R\to\infty} \frac{3R}{|z^2-R^2|} = 0 \qquad \forall z$$

$$\Rightarrow \quad f \equiv 0$$

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