Complex Analysis 3/28

Recall (Fundamental Thm of Alg, Thm 5.12)

Every nonconstant polynomial with complex coefficients has a zero in $\mathbb{C}$.

Remark

① Recall that given polynomials $f(z), g(z) \in \mathbb{C}[z]$, $\exists!\ q(z), r(z) \in \mathbb{C}[z]$ s.t.

division of poly

- $f(z) = q(z) \cdot g(z) + r(z)$
- $r(z) = 0$ or $\deg(r) < \deg(g)$

We say g is a factor of f, denoted $g \mid f$, if $r = 0$.

② $z - \alpha$ is a factor of a poly $f(z)$

$\iff f(\alpha) = 0$

③ α is called a zero of multiplicity k of $f(z)$ if $(z-\alpha)^k \mid f(z)$, $(z-\alpha)^{k+1} \nmid f(z)$

$\iff f(\alpha) = f'(\alpha) = \cdots = f^{(k-1)}(\alpha) = 0$, $f^{(k)}(\alpha) \neq 0$

(e.g. $f(z) = z^2$, 0 is a zero of multiplicity 2

$f'(z) = 2z$, $f''(z) = 2$. $f(0) = f'(0) = 0$, $f''(0) = 2 \neq 0$)

④ Suppose

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \quad a_n \neq 0$$

By $\underline{\text{Thm 5.12}}$, $\exists\ z_1 \in \mathbb{C}$ s.t. $f(z_1) = 0$

$$\overset{②}{\Rightarrow} \quad \overset{\text{deg: } n}{f(z)} = \overset{1}{(z-z_1)}\, \overset{n-1}{g_1(z)} \quad \text{for some } \overset{n-1}{g_1} \in \mathbb{C}[z]$$

Apply $\underline{\text{Thm 5.12}}$ to $g_1(z)$ and $g_j(z)$:

deg:

$$f(z) = \overset{1}{(z-z_1)}\, \overset{n-1}{g_1(z)}$$

$$= \overset{1}{(z-z_1)}\, \overset{1}{(z-z_2)}\, \overset{n-2}{g_2(z)}$$

$$= \cdots$$

by comparing coefficients

$$= (z-z_1)(z-z_2)\cdots(z-z_n)\cdot \overset{a_n}{\overset{\shortparallel}{C}}$$

multiplicities →

$$= a_n (z-\alpha_1)^{m_1} \cdots (z-\alpha_k)^{m_k}$$

← combine same z_j

Thus,

(i) a polynomial in $\mathbb{C}[z]$ of degree $n \geq 1$ has n zeros in $\mathbb{C}$ counting multiplicities

(ii) by comparing coefficients, one can get relations between zeros and coefficients such as $\displaystyle\sum_{k=1}^{n} z_k = \frac{-a_{n-1}}{a_n}$

⑤ Recall that people extend $\mathbb{R}$ to $\mathbb{C}$ because $x^2+1=0$ has NO zero in $\mathbb{R}$.

Thm 5.12 $\Rightarrow$ one DONOT need a further extension of $\mathbb{C}$ when solving polynomial equations.

Ch 6-7 Further properties of analytic functions

Uniqueness of analytic functions

Recall

- (Thm 2.12) Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$. If $\exists$ nonzero seq $\{z_k\}$ s.t. ① $\lim_{k\to\infty} z_k = 0$ ② $f(z_k) = 0 \ \forall k$, then $f \equiv 0$

- (Thm 6.5) If f is analytic in $D(\alpha; r)$, then $\exists \ c_n \in \mathbb{C}$ s.t. $f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n \quad \forall z \in D(\alpha; r)$

- (Advanced Calculus) Let D be a connected set. If ① $D = A \cup B$, ② A, B are open in D, ③ $A \cap B = \phi$, then $\begin{cases} A = \phi \\ B = D \end{cases}$ or $\begin{cases} A = D \\ B = \phi \end{cases}$.

e.g. $D = \underbrace{[0,1]}_{\text{open}} \cup \underbrace{[2,3]}_{\text{open}} \Rightarrow D$ is NOT connected

Uniqueness Thm (Thm 6.4)

i.e. open set connected

Suppose ① f is analytic in a region D. and ② $f(z_n)=0$ where $\{z_n\}$ is a seq. of distinct points and ③ $z_n \to z \in D$.

Then $f \equiv 0$ in D.

pf

$f(\lim_{n\to\infty} \omega_n) = \lim_{n\to\infty} f(\omega_n)$

continuity of f $\Rightarrow$ $f(z)=0 \quad \lim_{n\to\infty} 0 = 0$

Let

$$A = \{ z \in D : \exists \text{ distinct } \omega_n \in D \text{ s.t. } f(\omega_n)=0.\ \omega_n \to z \}$$

$$B = D \setminus A$$

Then ① $A \cup B = D$, $A \cap B = \phi$

assumption $\Rightarrow A \neq \phi$

② $\forall \alpha \in A\ \exists\ \varepsilon > 0\ \exists\ c_n \in \mathbb{C}$ s.t.

$$f(z) = \sum_{k=0}^{\infty} c_k (z-\alpha)^k \quad \forall z \in D(\alpha;\varepsilon) \subseteq D$$

open

By def of A, $\exists$ distinct $\omega_n \in D$ s.t. $\omega_n \to \alpha$ and $f(\omega_n) = 0$

By the uniqueness thm of power series, $c_n = 0\ \forall n$, i.e., $f \equiv 0$ in $D(\alpha;\varepsilon)$

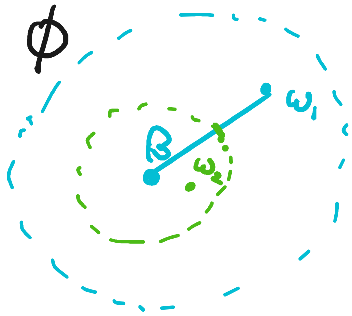
$\Rightarrow D(\alpha;\varepsilon) \subseteq A$

$\Rightarrow D(\alpha; \varepsilon) \subseteq A$

$\Rightarrow A$ is open

③ Given $\beta \in B$, claim $\exists\, \varepsilon > 0$ s.t. $D(\beta; \varepsilon) \subseteq B$

If not, $\forall\, \varepsilon > 0$, $D(\beta; \varepsilon) \setminus B \neq \phi$



$\Rightarrow \exists\, \omega_1 \in \underbrace{D(\beta; \varepsilon_0)} \setminus B$ $\leftarrow$ $D(\beta; \varepsilon_0) \subseteq D$

$\exists\, \omega_2 \in D(\beta; \frac{1}{2}|\omega_1 - \beta|) \setminus B$

$\vdots$

$\exists\, \omega_{n+1} \in D(\beta; \frac{1}{2}|\omega_n - \beta|)$

$\vdots$

$\Rightarrow \omega_n \neq \omega_m \in D$ and $\omega_n \to \beta$
$\forall\, n \neq m$

$\Rightarrow \beta \in A$ $(\rightarrow\leftarrow)$

So $\exists\, \varepsilon > 0$ s.t. $D(\beta; \varepsilon) \subseteq B \Rightarrow B$ is open.

Since D is connected, ①~③ $\Rightarrow A = D$

$\Rightarrow f \equiv 0$ in D #

if $D =$

we have $f \equiv 0$ $\equiv 1$

Cor 6.10

If two functions f and g, analytic in a region D, agree at a set of points with an accumulation point in D, then $f \equiv g$ in D

pf: Apply Thm 6.9 to $f-g$. #

Remark

$\sin(1/z) = 0$ on the set $\left\{ \frac{1}{n\pi} : n = \pm 1, \pm 2, \dots \right\}$ which has an acc point 0 but $\sin(1/z) \not\equiv 0$!!

This can happen because $\sin(1/z)$ is NOT analytic at 0

$\Rightarrow$ The hypotheses of Thm 6.9 are NOT satisfied

Applications of uniqueness thm:

① Prove functional equations such as

$$e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$$

pf

Given any fixed $z_2 \in \mathbb{C}$, $e^{z_1+z_2}$ and $e^{z_1}e^{z_2}$ are entire functions and

$$e^{x+\overset{x_2+iy_2}{z_2}} = e^x e^{z_2} \qquad \forall x \in \mathbb{R}$$

$= e^{x+x_2}(\cos y_2 + i\sin y_2) = e^x \cdot e^{x_2}(\cos y_2 + i \sin y_2)$

Since $\mathbb{R}$ has an acc point in $\mathbb{C}$, we have

$$e^{z_1+z_2} = e^{z_1}e^{z_2} \qquad \forall z_1 \in \mathbb{C}$$

#

② Thm 6.11

i.e. $\forall M>0$, $\exists B>0$ s.t. $|f(z)|>M$ $\forall |z|>B$

If f is entire and if $f(z)\to\infty$ as $z\to\infty$, then f is a polynomial.

Remark $f(z)=e^z \not\to \infty$

Note: $|e^{iy}| = |\cos y + i\sin y| = 1 \not\to \infty$ as $|y|\to\infty$

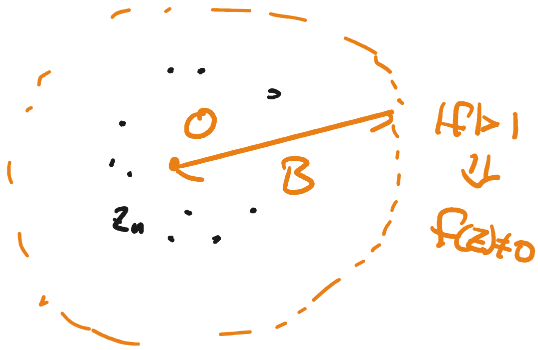
pf

(take $M=1$ in def)

By assumption, $\exists B>0$ s.t. $|z|>B \Rightarrow |f(z)|>1$

Claim

f has finite zeros



pf

If not, $\exists\ |z_n|\le B$, $n=1,2,\cdots$, ($z_n\neq z_m$ $\forall n\neq m$)

s.t. $f(z_n)=0$

Since $\overline{D(0;B)}$ is compact, $\{z_n\}$ has an acc pt in $\overline{D(0;B)} \subseteq \mathbb{C}$

uniqueness thm $\Rightarrow$ $f\equiv 0$ in $\mathbb{C}$ $(\rightarrow\leftarrow)$ #

Let $\alpha_1,\cdots,\alpha_N$ be the zeros of f, ($|\alpha_j|\le B$) and let k_j be the multiplictiy of α_j, i.e.

$$f(\alpha_j)=f^{(1)}(\alpha_j)=\cdots=f^{(k_j-1)}(\alpha_j)=0$$

$$f^{(k_j)}(\alpha_j)\neq 0$$

Claim

$k_j<\infty$

Pf

If not, $f(\alpha_j) = f^{(1)}(\alpha_j) = \cdots = 0$

$D(\alpha_j;\infty)$

f is entire

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{\overset{=0}{f^{(n)}(\alpha_j)}}{n!}(z-\alpha_j)^n \quad \forall z \in \mathbb{C}$$

$$= 0 \qquad \forall z \in \mathbb{C} \ (\rightarrow\leftarrow)_{\#}$$

Will finish on Thursday ....