# Complex Analysis 3/24

**Recall**

- Suppose $f$ is analytic in $D(\alpha; r)$, $r \in (0, \infty]$. Then

  (i) for $\rho \in (0, r)$, $|a - \alpha| < \rho$, $C_\rho : a + \rho e^{i\theta}$, $0 \le \theta \le 2\pi$

$$f(a) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{z-a}\, dz$$

( Cauchy Integral Formula )

  (ii) $f^{(n)}(\alpha)$ exist and

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!} (z-\alpha)^n \qquad \forall z \in D(\alpha; r)$$

( Taylor expansion of analytic function )

- ( Prop 5.8 / Prop 6.7 )

  If $f$ is analytic in an open set $U \subseteq \mathbb{C}$ and $a \in U$, then

$$g(z) := \begin{cases} \dfrac{f(z) - f(a)}{z - a} & \text{if } z \in U \setminus \{a\} \\ f'(a) = \lim\limits_{z \to a} \dfrac{f(z) - f(a)}{z - a} & \text{if } z = a \end{cases}$$

  is also analytic in $U$

**Cor 5.9**

Suppose $f$ is analytic in $U \underset{\text{open}}{\subseteq} \mathbb{C}$, $\overbrace{a_1, \cdots, a_N}^{\text{distinct}} \in U$ and $f(a_k) = 0$, $k = 1, \cdots, N$. Then

$$g(z) \doteq \begin{cases} \dfrac{f(z)}{(z-a_1)(z-a_2)\cdots(z-a_N)}, & \forall z \in U - \{a_1, \cdots, a_N\} \\ \lim\limits_{z \to a_k} \dfrac{f(z)}{(z-a_1)\cdots(z-a_N)} & \text{if } z = a_k, \ k = 1, \cdots, N \end{cases}$$

is analytic in $U$.

pf (induction)

Let $f_0(z) = f(z)$, and

$$f_1(z) := \begin{cases} \dfrac{f_0(z) - f_0(a_1)}{z - a_1} = \dfrac{f_0(z)}{z-a_1}, & z \in U, \ z \neq a_1 \\ f_0'(a_1) = \lim\limits_{z \to a_1} \dfrac{f_0(z)}{z-a_1}, & z = a_1 \end{cases}$$

$f_0(a_1) = 0$

Then by Prop 5.8, $f_1$ is analytic in $U$, and $f_1(a_k) = \dfrac{f_0(a_k)}{a_k - a_1} = 0 \quad \forall k = 2, \cdots, N.$

Inductively, let

$$f_k(z) := \begin{cases} \dfrac{f_{k-1}(z) - f_{k-1}(a_k)}{z - a_k} = \dfrac{f_{k-1}(z)}{z-a_k}, & z \neq a_k, \ z \in U \\ f_{k-1}'(a_k) = \lim\limits_{z \to a_k} \dfrac{f_{k-1}(z)}{z-a_k}, & z = a_k \end{cases}$$

By Prop 5.8, $f_k$ is analytic in $U$, and

$f_k(a_l) = 0, \quad l = k+1, \cdots, N$

So we have $g(z) = f_N(z)$ is analytic in $U$.

**Remark** (§6.2, p81)


Let $U \subseteq \mathbb{C}$ be open.

If $f : U \to \mathbb{C}$ is analytic and $D(\alpha; r) \subseteq U$, then $\exists\, c_n \in \mathbb{C}$ s.t.

$$f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n \qquad \forall z \in D(\alpha; r)$$

Note that "$\forall z \in D(\alpha;r)$" CANNOT be replaced by "$\forall z \in U$"

**Example** (p. 81-82)

(i) $f(z) = \dfrac{1}{z-1}$ is analytic in $U = \mathbb{C} \setminus \{1\}$

In particular, $f$ is analytic at $z=2$.

Furthermore,

$$f(z) = \frac{1}{z-1} = \frac{1}{(z-2)+2-1} = \frac{1}{1+(z-2)}$$

$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$

$$= \sum_{n=0}^{\infty} (-1)^n (z-2)^n \qquad \forall\, |z-2| < 1$$

i.e. $z \in D(2;1)$

Taylor expansion around $z=2$

But this equality is NOT true for $|z-2| > 1$ (so ~~$\forall z \in U$~~ )

Also note that

Taylor expansion around $z=0$

$$f(z) = \frac{1}{z-1} = \frac{-1}{1-z} = \sum_{n=0}^{\infty} -z^n \quad \forall\ |z|<1$$

Different expansions near different points!!

(ii) Find a power series reprentation for $\frac{1}{z^2}$ near 3:

$$\frac{1}{z^2} = \left(\frac{1}{3+z-3}\right)^2 = \frac{1}{9}\left(\frac{1}{1+\frac{z-3}{3}}\right)^2$$

$$= \frac{1}{9}\left(\sum_{n=0}^{\infty}\left(\frac{-1}{3}\right)^n (z-3)^n\right)^2$$

$(\sum a_n z^n)(\sum b_n z^n)$
$= \sum_{n=0}^{\infty}(\sum_{k=0}^{n} a_k b_{n-k}) z^n$
if $|z|<$ both radii of convergence

$$= \sum_{n=0}^{\infty} \frac{1}{9}\left(\sum_{k=0}^{n} \frac{(-1)^k}{3^k}\frac{(-1)^{n-k}}{3^{n-k}}\right)(z-3)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{9}\frac{n+1}{3^n}(z-3)^n$$

for $|z-3| < \dfrac{1}{\overline{\lim_{n\to\infty}}\sqrt[n]{\frac{1}{9}\left(\frac{-1}{3}\right)^n}} = 3$ #

$= \overline{\lim_{n\to\infty}} \sqrt[n]{\frac{1}{9}} \cdot \sqrt[n]{\left|\frac{-1}{3}\right|^n}$

$= \overline{\lim_{n\to\infty}} \sqrt[n]{\frac{1}{9}} \cdot \frac{1}{3}$ 1

$= \frac{1}{3}$

## Fundamental Thm of Algebra

## Liouville Thm (Thm 5.10)

A bounded entire function is constant.
$\hat{f}$

pf

Let $a, b \in \mathbb{C}$, and let $M$ be an upper bound of $|f(z)|$, $z \in \mathbb{C}$.

Then, by Cauchy Integral Formula, $\forall\ R > \max\{|a|, |b|\}$,

$C_R: Re^{i\theta}$, $0 \le \theta \le 2\pi$

$$\underbrace{|f(a) - f(b)|}_{\text{indep of } R}$$

$$= \left| \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{z-a} dz - \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{z-b} dz \right|$$

$$= \left| \frac{1}{2\pi i} \int_{C_R} \frac{f(z)(a-b)}{(z-a)(z-b)} dz \right|$$

$$\le \underbrace{2\pi R}_{= \text{length}(C_R)} \cdot \frac{1}{2\pi} \frac{M \cdot (a-b)}{(R-|a|)(R-|b|)} \to 0 \text{ as } R \to \infty$$

$|z-a| \ge |z| - |a| = R - |a|$ $\quad \forall z \in C_R$

$|z-b| \ge |z| - |b| = R - |b|$

So $f(a) = f(b)$ $\quad \forall a, b \in \mathbb{C}$. #

Extended Liouville Thm (Thm 5.11)

Extended Liouville

If $f$ is entire and if, for some $k \geq 0$, there exist constants $A, B \geq 0$ s.t.

$$|f(z)| \leq A + B|z|^k \quad \forall z \in \mathbb{C},$$

then $f$ is a polynomial of degree at most $k$.

pf (induction on $k$)

The case $k=0$ is the original Liouville Thm

Assume the theorem is true for $k-1$.

If $|f(z)| \leq A + B|z|^k \ \forall z \in \mathbb{C}$, then $(*)$

by Prop 5.8, the function

$$g(z) = \begin{cases} \dfrac{f(z)-f(0)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases}$$

is entire, and

$$|g(z)| = \frac{|f(z)-f(0)|}{|z|} \leq \frac{|f(z)|+|f(0)|}{|z|}$$

$$\overset{(*)}{\leq} \frac{A+B|z|^k+|f(0)|}{|z|} = \frac{A+|f(0)|}{|z|} + B|z|^{k-1}$$

$$\leq A+|f(0)| + B|z|^{k-1} \quad \forall |z|>1$$

By Continuity of $g(z)$, $\exists\, M$ s.t.

$$|g(z)| \leqslant M \qquad \forall\ |z| \leqslant 1$$

If we take

$$D = \max\{M,\ A + |f(0)|\}$$

$$E = B$$

then

$$|g(z)| \leqslant D + E|z|^{k-1} \qquad \forall z \in \mathbb{C}$$

induction $\Rightarrow$ hypothesis $g(z)$ is a polynomial of degree at most $k-1$

$\Rightarrow$ $f(z) = f(0) + g(z) \cdot z$ is a polynomial of degree at most $k$. #

<u>Lemma</u>

If $p(z)$ is a nonconstant polynomial, (i.e. $\deg p(z) \geq 1$)

then

$$p(z) \longrightarrow \infty \quad \text{as} \quad z \to \infty$$

That is, $\forall M > 0 \;\exists R > 0$ s.t.

$$|p(z)| > M \qquad \forall |z| > R.$$

pf (induction on $k = \deg p(z)$)

<u>$k = \deg p(z) = 1$:</u> $p(z) = az + b$, $a \neq 0$

$\forall M > 0$, $\exists R = \dfrac{M + |b|}{|a|} > 0$ s.t.

$$|p(z)| = |az+b| \geq |a||z| - |b| > \cancel{|a|}\frac{M+|b|}{\cancel{|a|}} - |b| = M$$

$$\forall |z| > R$$

<u>$k-1 \Rightarrow k$:</u> $p(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_0$, $a_k \neq 0$

By induction hypothesis, $\exists R_0 > 0$ s.t.

$$\circledast \quad |a_k z^{k-1} + a_{k-1} z^{k-2} + \cdots + a_1| > 1 \qquad \forall |z| > R_0$$

$\Rightarrow \forall M > 0$, $\exists R = \underbrace{|a_0| + M + R_0}_{\geq R_0} > 0$ s.t.

$$|p(z)| = |a_k z^k + \cdots + a_0|$$

$$\geq \underbrace{|a_k z^k + a_{k-1} z^{k-1} + \cdots + a_1 z|}_{= \underbrace{|a_k z^{k-1} + \cdots + a_1|}_{>1 \text{ by } \circledast} \cdot |z|} - |a_0|$$

$$> (\cancel{|a_0|} + M + R_0) - \cancel{|a_0|} > M$$

$$\forall |z| > R$$

This completes the induction. #

## Fundamental Theorem of Algebra (Thm 5.12)

Every nonconstant polynomial with complex coefficients has a zero in $\mathbb{C}$.

(i.e. $\mathbb{C}$ is algebraically closed)

pf

Suppose $p(z)$ is a nonconstant polynomial s.t. $p(z) \neq 0$ $\forall z \in \mathbb{C}$.

$\Rightarrow f(z) := \dfrac{1}{p(z)}$ is an entire function

By Lemma, $\exists R > 0$ s.t.

$$|p(z)| > 1 \qquad \forall |z| > R$$

$$\Rightarrow |f(z)| < 1 \qquad \forall |z| > R$$

By continuity of $f(z)$, $\exists M_0$ s.t.

$$|f(z)| \leq M_0 \qquad \forall |z| \leq R$$

So $|f(z)| \leq 1 + M_0 \qquad \forall z \in \mathbb{C}$

By Liouville Thm, $f(z)$ is constant

$\Rightarrow p(z)$ is also constant ($\rightarrow\leftarrow$)

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