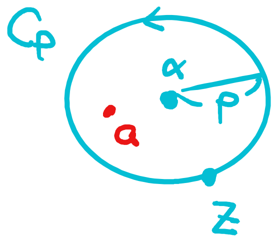
Complex Analysis 3/21

Recall

① **Lemma 5.4**

Let $\alpha, a \in \mathbb{C}$, $\rho > 0$. $C_\rho : \alpha + \rho e^{i\theta}$, $0 \le \theta \le 2\pi$

Suppose $|a - \alpha| < \rho$. Then



$$\int_{C_\rho} \frac{1}{z-a}\, dz = 2\pi i$$

② In the proof, we showed that

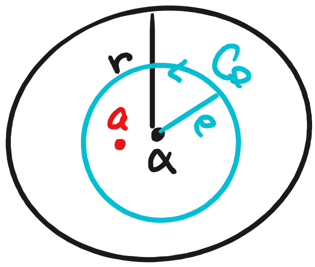
$$\frac{1}{z-a} = \frac{1}{z-\alpha} \cdot \frac{1}{1-\omega} = \sum_{n=0}^{\infty} \frac{\omega^n}{z-\alpha}$$

where $\omega = \dfrac{a-\alpha}{z-\alpha}$ ↑ converges uniformly

Cauchy Integral Formula (Thm 5.3, Thm 6.4)

Suppose f is analytic in $D(\alpha; r)$ for some $r \in (0, +\infty]$. Suppose $\rho \in (0, r)$ and $|a-\alpha| < \rho$

Then
$$\boxed{f(a) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{z-a}\, dz}$$



where C_ρ is the circle

$$\alpha + \rho e^{i\theta}, \quad 0 \le \theta \le 2\pi$$

pf

By Closed Curved Thm (Thm 4.16, Cor 5.2, Thm 6.3)

$$\int_{C_\rho} \frac{f(z)-f(\alpha)}{z-\alpha}\,dz = 0$$

$$\parallel$$

$$\int_{C_\rho} \frac{f(z)}{z-\alpha}\,dz - \int_{C_\rho} \frac{f(\alpha)}{z-\alpha}\,dz = 0$$

Lemma 5.4

$$\Rightarrow \int_{C_\rho} \frac{f(z)}{z-\alpha}\,dz = f(\alpha)\int_{C_\rho} \frac{1}{z-\alpha}\,dz = 2\pi i \cdot \boxed{f(\alpha)}$$

#

Taylor expansion of an analytic function

Thm 5.5 (Also see Thm 6.5 , Thm 6.6)

If f is analytic in $D(\alpha; r)$, $r \in (0, +\infty]$

then $f^{(n)}(\alpha)$ exists for $n = 1, 2, \cdots$, and

$$\boxed{f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(z-\alpha)^n}$$

$\forall\, z \in D(\alpha; r)$

pf

$0 \le \theta \le 2\pi$

Let $\rho \in (0, r)$ and C_ρ be the circle $\alpha + \rho e^{i\theta}$

By Cauchy Integral Formula, $\forall\, z \in D(\alpha; \rho)$

$$f(z) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(\omega)}{\omega - z}\,d\omega$$

Recall in the proof of Lemma 5.4, we proved

$$\sum_{n=0} \frac{(z-\alpha)^n}{(\omega-\alpha)^{n+1}} \longrightarrow \frac{1}{\omega-\alpha}\cdot\frac{1}{1-\frac{z-\alpha}{\omega-\alpha}} = \frac{1}{\omega-z}$$

uniformly on C_ρ (as functions ω)

$$\Rightarrow \quad f(z) = \frac{1}{2\pi i}\int_{C_\rho} f(\omega)\cdot\sum_{n=0}^{\infty}\frac{(z-\alpha)^n}{(\omega-\alpha)^{n+1}}\,d\omega$$

$$= \sum_{n=0}^{\infty}\underbrace{\left(\frac{1}{2\pi i}\int_{C_\rho}\frac{f(\omega)}{(\omega-\alpha)^{n+1}}\,d\omega\right)}_{\text{indep. of } z,\ \ "C_n \in \mathbb{C}}\cdot(z-\alpha)^n$$

$\forall z \in D(\alpha;\rho)$

So, by the uniqueness of power series ($\frac{\text{Cor } 2.11}{\text{Thm } 2.12}$)

$$C_n = \frac{f^{(n)}(\alpha)}{n!} = \frac{1}{2\pi i}\int_{C_\rho}\frac{f(\omega)}{(\omega-\alpha)^{n+1}}\,d\omega$$

Since ρ is arbitrary in $(0, r)$, we can conclude

$$f(z) = \sum_{n=0}^{\infty} C_n (z-\alpha)^n = \sum_{n=0}^{\infty}\frac{f^{(n)}(\alpha)}{n!}(z-\alpha)^n$$

$\forall\ z \in D(\alpha; r)$ #

<u>Cor 5.6</u> (Also see <u>Thm 6.8</u>)

Any analytic function is infinitely differentiable

pf

(by def)

If f is analytic at α, then $\exists\, r>0$ s.t. f is differentiable/analytic in $D(\alpha; r)$

Thm5.5 $\Rightarrow$ f is ∞ differentiable at α #

<u>Remark</u> (an idea about how good a function is)

Bad — arbitrary function ———— continuous function —

difference is big / difference is not so big

— real-valued C^∞ ———— "C-R eq" (complex) analytic — (complex) poly → Good

CANNOT be determined by its values on a non-dense subset

determined by its values on a set with an acc pt (<u>Cor 2.14</u>, <u>Cor 6.10</u>)

determined by its values on a finite set

<u>Example</u>

① Suppose f is entire and $f(\frac{1}{n}) = 0 \ \forall n$

(<u>Note</u>: $\{\frac{1}{n}\} \subseteq \overline{D(0;1)} \subsetneq \mathbb{C}$)

$\Rightarrow f \equiv 0$ (<u>Thm5.5</u> + <u>Thm 2.12</u>)

② $f(z) := \begin{cases} e^{\frac{-1}{(|z|-1)^2}}, & |z| > 1 \end{cases}$

, $|z| \le 1$

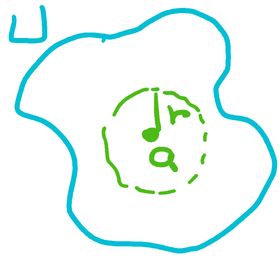
is real-valued C^∞ but $f(\frac{1}{n}) = 0 \quad \forall n$

<u>Prop 5.8</u> (Also see <u>Prop 6.7</u>)

If f is analytic in an open set U ($\subseteq \mathbb{C}$) and $a \in U$, then

$$g(z) := \begin{cases} \dfrac{f(z) - f(a)}{z-a} & \text{if } z \in U - \{a\} \\ f'(a) & \text{if } z = a \end{cases}$$

is also analytic in U



<u>pf</u>

g is clearly analytic in $U \setminus \{a\}$.

Suff. to show g is analytic at a:

Let $r > 0$ s.t. $D(a;r) \subseteq U$

Thm 5.5

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n \quad \forall z \in D(a;r)$$

$$\Rightarrow g(z) = \frac{1}{z-a} \left(\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n - f(a) \right) \quad \forall z \in D(a;r) - \{a\}$$

$$= \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^{n-1}$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n+1)}(a)}{(n+1)!} (z-a)^n$$

Note that at $z=a$,

$$g(a) = f'(a) = \frac{f^{(1)}(a)}{1!} = \sum_{n=0}^{\infty} \frac{f^{(n+1)}(a)}{(n+1)!}(a-a)^n$$

$$\Rightarrow \quad g(z) = \sum_{n=0}^{\infty} \frac{f^{(n+1)}(a)}{(n+1)!}(z-a)^n \qquad \forall\, z \in D(a;r)$$

$\Rightarrow$ g is analytic a #