Complex Analysis 3/17

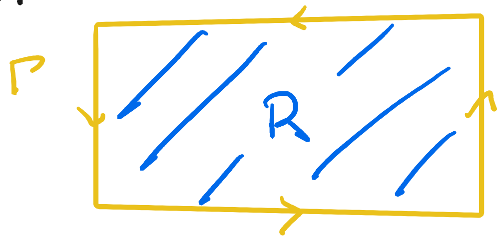
Recall

Rectangle Thm (Thm 4.14, Thm 5.1, Thm 6.1)

Suppose f is analytic on $U \underset{\text{open}}{\subseteq} \mathbb{C}$, and $R \subseteq U$ is a rectangle. Then

(i) $\int_\Gamma f(z)\,dz = 0$

(ii) $\int_\Gamma g(z)\,dz = 0$



where $\Gamma = \partial R$, $a \in U$, and

$$g(z) = \begin{cases} \dfrac{f(z) - f(a)}{z - a}, & z \in U \setminus \{a\} \\ f'(a), & z = a \end{cases}$$

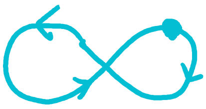
Closed curve thm

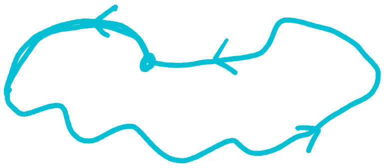
Def 4.13

A curve C given by $z(t)$, $a \le t \le b$, is called closed if $z(a) = z(b)$

A closed curve C is called a simple closed curve if $\nexists$ other points coincide, i.e.

$s \neq t$ $\quad z(s) = z(t) \Rightarrow \{s, t\} = \{a, b\}$

e.g. ①  is closed, NOT simple closed

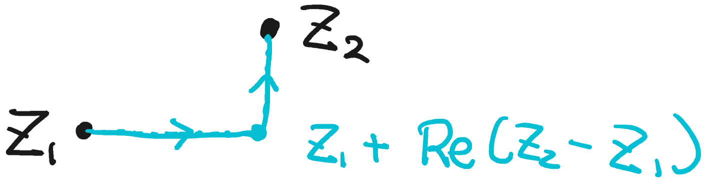
②  is simple closed

<u>Notations</u> ($z_j \in \mathbb{C}$, $r \overset{\in \mathbb{R}}{\geq} 0$)

① $D(z_0; r) := \{ z \in \mathbb{C} \mid |z - z_0| < r \}$

② $D(z_0; \infty) := \mathbb{C}$

③ $\int_{z_1}^{z_2} f(\zeta)\, d\zeta := \int_C f(\zeta)\, d\zeta$

where $C =$ 

<u>Integral Thm</u> (Thm 4.15, Cor 5.2, Thm 6.2)

Suppose $r \in (0, \infty]$. If

$$f : D(z_0; r) \longrightarrow \mathbb{C}$$

is analytic, then $\exists$ analytic functions F, G on $D(z_0; r)$ s.t.

a might not be z_0

$F'(z) = f(z)$, $\forall z \in D(z_0; r)$

$G'(z) = g(z)$ $(a \in D(z_0; r))$

where $\quad g(z) = \dfrac{f(z) - f(a)}{\ldots}$ $\quad z \in D(z_0; r) \setminus \{a\}$

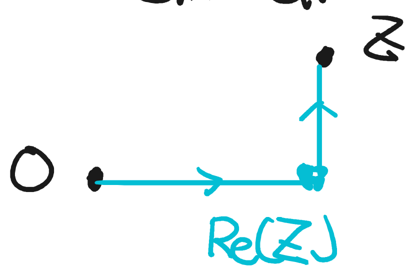
$$g(z) = \begin{cases} \dfrac{\quad\quad}{z-a} & z \in D(z_0, r)\setminus\{a\} \\ f'(a) & z = a \end{cases}$$

pf

We will construct $F(z)$ by Rectangle Thm.

The construction of $G(z)$ is similar

Special case: $z_0 = 0$



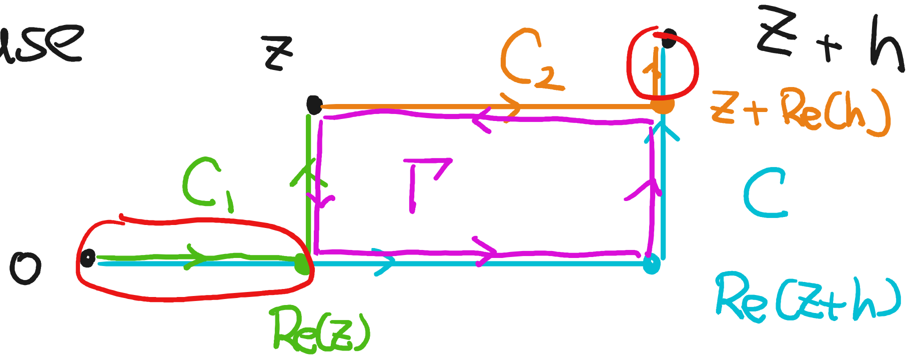
Let

$$F(z) := \int_0^z f(\zeta)\, d\zeta$$

Note that

$$F(z+h) = F(z) + \int_z^{z+h} f(\zeta)\, d\zeta$$

because



$$F(z+h) - \left(F(z) + \int_z^{z+h} f(\zeta)\, d\zeta\right)$$

$$= \int_C f(\zeta)\, d\zeta - \left(\int_{C_1} f(\zeta)\, d\zeta + \int_{C_2} f(\zeta)\, d\zeta\right)$$

Rectangle Thm

$$= \int_\Gamma f(\zeta)\, d\zeta = 0$$

Also note that

$$\int_z^{z+h} 1 \, d\zeta = \int_z^{z+h} \frac{d}{d\zeta}(\zeta) \, d\zeta$$

Prop 4.12 (fundamental thm of Calculus for line integral)

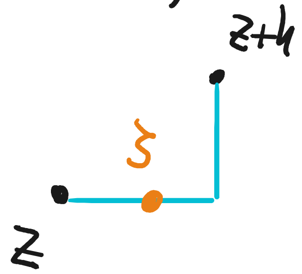
$$= \zeta \Big|_z^{z+h} = h$$

$f(z) \cdot \frac{1}{h} \int_z^{z+h} 1 \, d\zeta$ =

$$\Rightarrow \frac{F(z+h) - F(z)}{h} - f(z)$$

$$= \frac{1}{h} \left(\int_z^{z+h} f(\zeta) \, d\zeta - \int_z^{z+h} f(z) \, d\zeta \right)$$

$$= \frac{1}{h} \int_z^{z+h} f(\zeta) - f(z) \, d\zeta$$



Finally, since f is continuous at z, $\forall \varepsilon > 0 \ \exists \delta$ s.t.

$$|f(\zeta) - f(z)| < \varepsilon \quad \forall \ |\zeta - z| < \delta$$

$\Rightarrow \forall \ |h| < \delta$,

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right|$$

$$= \frac{1}{|h|} \left| \int_z^{z+h} f(\zeta) - f(z) \, d\zeta \right|$$

M-L

length of

inequality → $\le \frac{1}{|h|}\cdot \varepsilon \cdot (|\mathrm{Re}(h)| + |\mathrm{Im}(h)|)$

$$\le \frac{1}{|h|}\cdot \varepsilon \cdot 2|h| = 2\varepsilon$$

$$\Rightarrow \quad F'(z) = \lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = f(z)$$

<u>General case</u> : any z_0 , $f: D(z_0;r) \to \mathbb{C}$

Let $h(z) = f(z+z_0)$

$\Rightarrow$ $h: D(0;r) \to \mathbb{C}$

special case $\Rightarrow$ $\exists\, H : D(0;r) \to \mathbb{C}$ s.t. $H'(z) = h(z)$

$\Rightarrow$ Let $F(z) = H(z-z_0)$

$\Rightarrow$ $F: D(z_0;r) \to \mathbb{C}$ analytic and $F'(z) = f(z)$

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<u>Closed Curve Thm</u> (Thm 4.16, Cor 5.2, Thm 6.3)

Suppose $r \in (0,\infty]$. If $f: D(z_0;r) \to \mathbb{C}$ is analytic, then for any closed, piecewise C^1 curve C in $D(z_0;r)$,

(i) $\int_C f(z)\,dz = 0$

(ii) $\int_C g(z)\,dz = 0$

where $g(z) = \begin{cases} \dfrac{f(z)-f(a)}{z-a}, & z \in D(z_0;r) \end{cases}$

$\llcorner \quad f'(a) \quad , \quad z = a$

$a \in D(z_0; r)$

pf

By Integral Thm, $\exists$ analytic

$G, F : D(z_0; r) \to \mathbb{C}$ s.t. $F'(z) = f(z)$

resp. $G'(z) = g(z)$

$\Rightarrow$ by Prop 4.12 (fund. thm. of Calculus for line integral)

$$\int_C f(z)\,dz = \int_C F'(z)\,dz$$

$$= F(z(b)) - F(z(a)) = 0$$

$\#$

e.g.

① $$\int_{C:\, e^{i\theta},\, 0 \le \theta \le 2\pi} z \, dz = \int_0^{2\pi} e^{i\theta} \cdot (i \cdot e^{i\theta})\, d\theta$$

$$= i \int_0^{2\pi} e^{2i\theta}\, d\theta = \frac{1}{2} e^{2i\theta} \Big|_{\theta=0}^{2\pi}$$

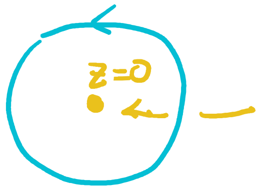
$$= \frac{1}{2}(1 - 1) = 0$$

② $\int \frac{1}{\ } dz \quad \int^{2\pi}$ [illegible]

$$\int_{C} \frac{1}{z} \, dz = \int_0^{2\pi} \frac{1}{\cancel{e^{i\theta}}} \cdot (i \cdot \cancel{e^{i\theta}}) \, d\theta$$

$C: e^{i\theta}, \ 0 \le \theta \le 2\pi$

$$= \int_0^{2\pi} i \, d\theta = 2\pi i \neq 0 \ !!$$



$\frac{1}{z}$ is NOT analytic at $z=0$

Ch5 Cauchy integral formula

Recall (from Advanced Calculus)

① (p.15) Let $D \subseteq \mathbb{C}$ and $f, f_n : D \to \mathbb{C}$, $n = 1, 2, \ldots$

be a seq of functions

We say $\{f_n\}_{n=1}^{\infty}$ converges to f uniformly in D if $\forall \varepsilon > 0$

$\exists N$ s.t. ← indep of $z \in D$

$$|f_n(z) - f(z)| < \varepsilon \quad \forall n > N, \ \forall z \in D$$

We say $\sum_{n=1}^{\infty} f_n$ converges uniformly in D

if $\{\sum_{k=1}^{n} f_k\}_{n=1}^{\infty}$ converges uniformly in D

② If $f_n \to f$ uniformly and f_n are continuous in D
then f is continuous in D

③ <u>Weierstrass M-test</u> (<u>Thm 1.9</u>, p15)
Suppose $|f_n(z)| \leq \underline{M_n} \leftarrow \mathbb{R}$ $\forall z \in D$, $n = 1, 2, \ldots$
If $\sum_{n=1}^{\infty} M_n < \infty$, then
$\sum_{n=1}^{\infty} f_n$ converges uniformly in D

<u>Cor</u>
If $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ has radius of convergence $R>0$
then, $\forall\, r<R$, $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ converges
uniformly in $\overline{D(z_0; r)}$

④ If a seq of continuous functions
$f_n : [a,b] \to \mathbb{R}$
converges uniformly in $[a,b]$, then

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b \lim_{n\to\infty} f_n(x)\, dx$$

<u>Prop 4.11</u> (line-integral ver. of ④)

Suppose $\{f_n\}$ is a seq. of continuous functions and $f_n \to f$ uniformly on a piecewise C^1 curve C. Then

$$\lim_{n\to\infty} \int_C f_n(z)\,dz = \int_C f(z)\,dz$$

pf

$\forall \varepsilon > 0 \ \exists N$ s.t. $|f_n(z) - f(z)| < \varepsilon \quad \begin{matrix} \forall z \in C \\ \forall n > N \end{matrix}$

$$\Rightarrow \left| \int_C f_n(z)\,dz - \int_C f(z)\,dz \right|$$

$$= \left| \int_C f_n(z) - f(z)\,dz \right|$$

M-L inequality

$$\leq \varepsilon \cdot \text{length}(C) \qquad \forall n > N$$

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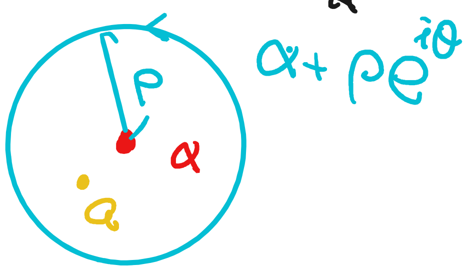

__Cauchy integral formula__

__Lemma 5.4__

Let $\alpha, a \in \mathbb{C}$, $\rho > 0$, and C_ρ be the circle $\alpha + \rho e^{i\theta}$, $0 \leq \theta \leq 2\pi$. Suppose $|a - \alpha| < \rho$. Then

$$\int_{C_\rho} \frac{1}{z-a}\,dz = 2\pi i$$

pf

Note that (if $a = \alpha$)

$$\int_{C_\rho} \frac{1}{z-\alpha}\,dz = \int_0^{2\pi} \frac{i \cancel{\rho e^{i\theta}}}{\cancel{\alpha} + \cancel{\rho e^{i\theta}} - \cancel{\alpha}}\,d\theta$$

$$= \int_0^{2\pi} i\,d\theta = 2\pi i$$

and

$$\int_{C_\rho} \frac{1}{(z-\alpha)^{k+1}}\,dz = 0 \quad , \; k = 1, 2, 3, \ldots$$

$$= \int_0^{2\pi} \frac{i\rho e^{i\theta}}{(\rho e^{i\theta})^{k+1}}\,d\theta = \int_0^{2\pi} i \frac{1}{(\rho e^{i\theta})^k}\,d\theta$$

$$= \frac{i}{\rho}\int_0^{2\pi} e^{-ik\theta}\,d\theta = \frac{i}{\rho}\,\frac{1}{-ik}\,e^{-ik\theta}\Big|_{\theta=0}^{\theta=2\pi} = 0$$

Write

$$\frac{1}{z-a} = \frac{1}{(z-\alpha)-(a-\alpha)} = \frac{1}{z-\alpha}\cdot\frac{1}{1-\frac{a-\alpha}{z-\alpha}}$$

$$= \frac{1}{z-\alpha}\cdot\frac{1}{1-\omega}$$

where $\omega = \dfrac{a-\alpha}{z-\alpha}$ has a fixed modulus

$$|\omega| = \frac{|a-\alpha|}{\rho} < 1 \qquad \forall z \in C_\rho$$

Since

$$\sum_{n=0}^{\infty}\left|\frac{1}{z-\alpha}\cdot\omega^n\right| = \sum_{n=0}^{\infty}\frac{1}{\rho}\cdot\Big(\underbrace{\frac{|a-\alpha|}{\rho}}_{<1}\Big)^n < \infty,$$

by M-test

$$\frac{1}{z-\alpha}\cdot\frac{1}{1-\omega} = \sum_{n=0}^{\infty}\frac{1}{z-\alpha}\cdot\omega^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{z-\alpha}\left(\frac{a-\alpha}{z-\alpha}\right)^n$$

Converges to $\frac{1}{z-a}$ uniformly on C_ρ

$$\Rightarrow \int_{C_\rho} \frac{1}{z-a}\,dz = \int_{C_\rho} \sum_{n=0}^{\infty} \frac{1}{z-\alpha}\left(\frac{a-\alpha}{z-\alpha}\right)^n dz$$

$$= \sum_{n=0}^{\infty} \int_{C_\rho} \frac{(a-\alpha)^n}{(z-\alpha)^{n+1}}\,dz = \sum_{n=0}^{\infty} (a-\alpha)^n \int_{C_\rho} \frac{1}{(z-\alpha)^{n+1}}\,dz$$

$$= (a-\alpha)^0 \cdot 2\pi i = 2\pi i$$

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