Complex Analysis 3/10

Reminders from TA

① Cauchy - Riemann eq:

$f = u + iv$ is analytic in U open $\mathbb{C}$

$\Leftrightarrow$ (i) f_x, f_y exist

(ii) f_x, f_y are continuous in U

(iii) $f_y = i f_x$ i.e. $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

e.g. $f(z) = \bar{z} = x - iy$ is NOT analytic

because $f_y = -i \neq i = i f_x$

② Power series:

$\sum_{n=0}^{\infty} a_n$ is absolutely convergent if

$\sum_{n=0}^{\infty} |a_n|$ is convergent

Prop

- absolutely converge $\Rightarrow$ converge
- For $|z| < R$ = radius of convergence of $\sum c_n z^n$

  $\sum_{n=0}^{\infty} c_n z^n$ is absolutely convergent.

e.g. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^n$ converges $\forall z \in \mathbb{C}$

Recall

① Fundamental Thm of Calculus for line integrals

- for piecewise C^1 curve $\gamma : [a,b] \to \mathbb{C}$,

$$\int_a^b \gamma'(t)\,dt = \gamma(b) - \gamma(a)$$

pf

Let $\gamma(t) = \gamma_1(t) + i\,\gamma_2(t)$

If γ is C^1, then

$$\int_a^b \gamma'(t)\,dt = \int_a^b \gamma_1'(t) + i\,\gamma_2'(t)\,dt$$

$$= \int_a^b \gamma_1'(t)\,dt + i\int_a^b \gamma_2'(t)\,dt$$

$$= \gamma_1(b) - \gamma_1(a) + i\,(\gamma_2(b) - \gamma_2(a))$$

$$= \gamma(b) - \gamma(a)$$

If γ is piecewise C^1, $\exists\ \underset{\underset{t_0}{\shortparallel}}{a} < t_1 < \cdots < t_{n-1} < \underset{\underset{t_n}{\shortparallel}}{b}$

s.t. γ is C^1 on each $[t_j, t_{j+1}]$.

$$\Rightarrow \int_a^b \gamma'(t)\,dt = \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \gamma'(t)\,dt$$

$$= \sum_{j=0}^{n} \gamma(t_{j+1}) - \gamma(t_j)$$

by continuity of γ

$$= \gamma(b) - \cancel{\gamma(t_n)} + \cancel{\gamma(t_n)} - \cancel{\gamma(t_{n-1})} + \cancel{\cdots} - \gamma(a)$$

$$= \gamma(b) - \gamma(a) \qquad \#$$

- If F is analytic on $C: z(t),\ a \le t \le b$, then

$$\int_C F'(z)\,dz = F(z(b)) - F(z(a))$$

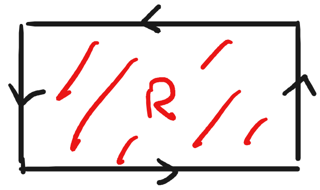
② <u>Lemma</u> (p.52)

If $f(z) = \alpha + \beta z$, $\alpha, \beta \in \mathbb{C}$, then

$$\int_\Gamma f(z)\,dz = 0$$

where

Γ:



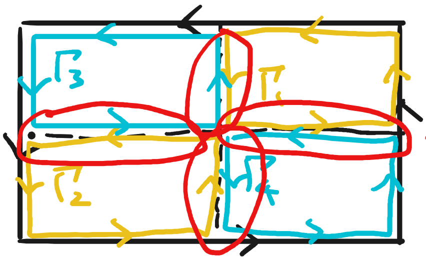
<u>Rectangle Thm</u> (<u>Thm 4.14</u>, <u>Thm 6.1</u>)

Suppose f is analytic on $U \underset{\text{open}}{\subseteq} \mathbb{C}$ and $R \subseteq U$

Then $$\int_\Gamma f(z)\,dz = 0$$

<u>pf</u>

Γ:



cancel

Let $I = \int_\Gamma f(z)\,dz$.

<u>Step 1</u>: Split R into 4 congruent subrectangles

By <u>Prop 4.7</u>,

$$I = \int_\Gamma f(z)\,dz = \sum_{j=1}^{4} \int_{\Gamma_j} f(z)\,dz$$

Let $\Gamma^{(1)}$ be the one s.t.

$$\left|\int_{\Gamma^{(1)}} f(z)\,dz\right| = \max\left\{\left|\int_{\Gamma_j} f(z)\,dz\right|\right\}$$

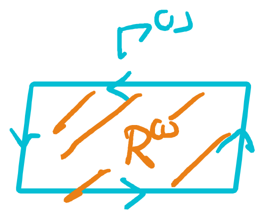
$$\left|\int_{\Gamma^{(1)}} f(z)\,dz\right| = \max_{j=1,\cdots,4}\left\{\left|\int_{\Gamma_j} f(z)\,dz\right|\right\}$$

Since

$$|I| = \left|\sum_{j=1}^{4}\int_{\Gamma_j} f(z)\,dz\right|$$

$$\leq \sum_{j=1}^{4}\left|\int_{\Gamma_j} f(z)\,dz\right|$$

$$\leq 4\left|\int_{\Gamma^{(1)}} f(z)\,dz\right|,$$



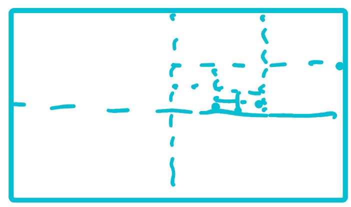
we have

$$\frac{|I|}{4} \leq \left|\int_{\Gamma^{(1)}} f(z)\,dz\right|.$$

Let $R^{(1)}$ be the rectangle bounded by $\Gamma^{(1)}$

<u>Step2</u> Repeat <u>Step1</u> to $R^{(1)}$, we get

$$R \supseteq R^{(1)} \supseteq R^{(2)} \supseteq \cdots$$



and their boundaries

$$\Gamma,\ \Gamma^{(1)},\ \Gamma^{(2)},\ \cdots$$

s.t. ① $\operatorname{diam} R^{(k+1)} = \frac{1}{2}\operatorname{diam} R^{(k)}$

② $\left|\int_{\Gamma^{(k)}} f(z)\,dz\right| \geq \frac{|I|}{4^k}$

<u>Step3</u>

Since each $R^{(k)}$ is compact and nonempty, by the nested property of compact sets,

$$\exists \; z_0 \in \bigcap_{k=1}^{\infty} R^{(k)}$$

Let

$$\varepsilon_z = \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)$$

Then

$$f(z) = f(z_0) + f'(z_0)\cdot(z - z_0) + \varepsilon_z \cdot (z - z_0)$$

and, by the analyticity of f at z_0,

$$\varepsilon_z \longrightarrow 0 \quad \text{as} \quad z \to z_0 .$$

Step 4

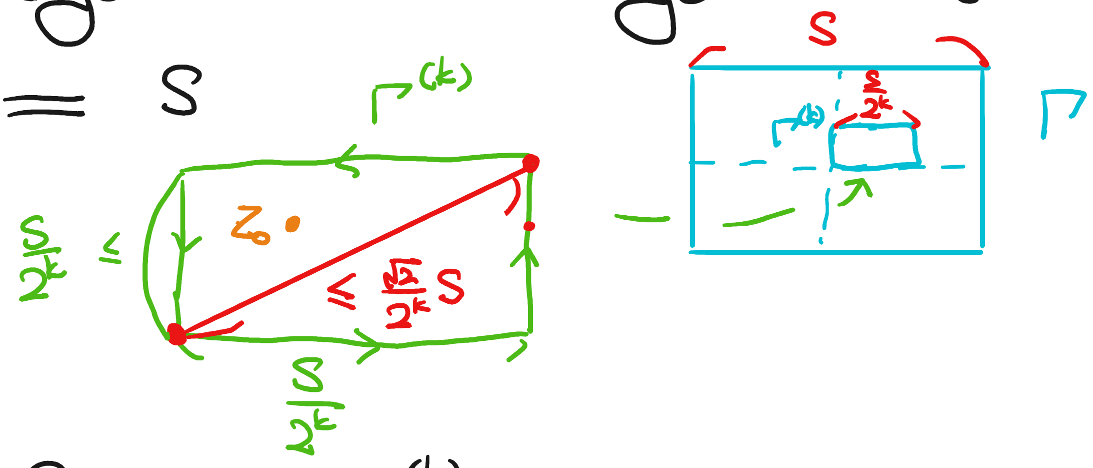
By Lemma,

$$\int_{\Gamma^{(k)}} f(z)\,dz$$

$$\overset{\text{Step 3}}{=} \int_{\Gamma^{(k)}} \Big(\overbrace{f(z_0) + f'(z_0)\cdot(z - z_0)}^{\text{linear}} + \varepsilon_z \cdot (z - z_0) \Big)\, dz$$

$$\overset{\text{Lemma}}{=} \int_{\Gamma^{(k)}} \varepsilon_z \cdot (z - z_0)\, dz$$

Let the length of the largest side of $\Gamma = S$



Note that for $z \in \Gamma^{(k)}$,

$$|z - z_0| \leq \frac{\sqrt{2}}{2^k} S$$

<u>Step 5</u>

$\forall \, \varepsilon > 0$, $\exists \, N$ s.t. $\quad \because \varepsilon_z \to 0$ as $z \to z_0$

$$|\varepsilon_z - 0| < \varepsilon \qquad \forall \, |z - z_0| \leq \frac{\sqrt{2}}{2^N} S$$

$\Rightarrow \forall \, k \geq N$, by <u>Prop 4.10</u> (M-L formula),

$$\frac{|I|}{4^k} \leq \left| \int_{\Gamma^{(k)}} f(z) \, dz \right| = \left| \int_{\Gamma^{(k)}} \varepsilon_z \cdot (z - z_0) \, dz \right|$$

$$\leq \varepsilon \cdot \frac{\sqrt{2}}{2^k} S \cdot \text{length}(\Gamma^{(k)})$$

an upper bound of $|\varepsilon_z \cdot (z - z_0)|$ on $\Gamma^{(k)}$

$$\leq \varepsilon \cdot \frac{\sqrt{2}}{2^k} S \cdot 4 \cdot \frac{S}{2^k} = \varepsilon \cdot \frac{1}{4^k} \cdot 4\sqrt{2} \, S^2$$

$\Rightarrow \forall \varepsilon > 0 \;\; \exists N$ s.t. $\forall \, k \geq N$,

$$\frac{|I|}{\cancel{4^k}} \le \varepsilon \cdot \frac{4\sqrt{2}}{\cancel{4^k}} S^2$$

independent of N, k

$$\Leftrightarrow \quad |I| \le \varepsilon \cdot 4\sqrt{2}\, S^2$$

So $\quad |I| = 0 \quad \Rightarrow \quad I = \int_\Gamma f(z)\,dz = 0$

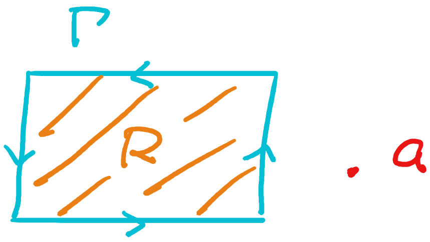
#

<u>Rectangle Thm II</u> (<u>Thm 5.1</u> , <u>Thm 6.1</u>)

Suppose f is analytic on $U \underset{\text{open}}{\subseteq} \mathbb{C}$ and $R \subseteq U$. For $a \in U$, let

$$g(z) := \begin{cases} \dfrac{f(z) - f(a)}{z - a} & z \in U \setminus \{a\} \\ f'(a) & z = a \end{cases}$$

Then $\quad \int_\Gamma g(z)\,dz = 0$



<u>pf</u>

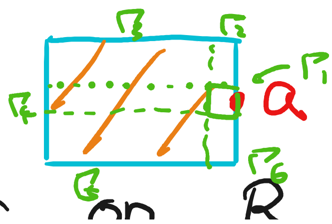
<u>case1</u> $\quad a \notin R$

Note that $\dfrac{1}{z-a}$ is analytic on $\mathbb{C} \setminus \{a\} \supseteq R$

$\Rightarrow \quad \dfrac{f(z)-f(a)}{z-a}$ is analytic on $U \setminus \{a\} \supseteq R$

Rectangle Thm

$\Rightarrow \quad \int_\Gamma \dfrac{f(z)-f(a)}{z-a}\,dz = 0$



<u>case2</u> $\quad a \in \Gamma = \partial R$

Note that g is continuous on R

Note that g is continuous on R $\leftarrow$ compact

$\Rightarrow \exists M > 0$ s.t. $|g(z)| \le M \quad \forall z \in R$

We divide R into 6 subrectangle

$$\Rightarrow \int_{\Gamma} g(z)\,dz = \sum_{j=1}^{6} \int_{\Gamma_j} g(z)\,dz$$

$$\overset{\text{by case1}}{=} \int_{\Gamma_i} g(z)\,dz$$

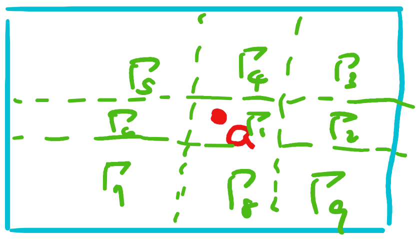
Note that $\forall \varepsilon > 0$, $\exists$ division s.t.

$$\text{length}(\Gamma_i) < \varepsilon/M$$

$$\Rightarrow \left| \int_{\Gamma} g(z)\,dz \right| = \left| \int_{\Gamma_i} g(z)\,dz \right|$$

$$\overset{\text{M-L}}{\le} M \cdot \varepsilon/M = \varepsilon \quad \forall \varepsilon > 0$$

$$\Rightarrow \int_{\Gamma} g(z)\,dz = 0$$



<u>case3</u> $a \in \text{int}(R)$

Similar as <u>case 2</u>,

$\forall \varepsilon > 0$ $\exists$ division s.t. $\text{length}(\Gamma_i) < \varepsilon/M$

$$\Rightarrow \left| \int_{\Gamma} g(z)\,dz \right| = \left| \sum_{j=1}^{9} \int_{\Gamma_j} g(z)\,dz \right|$$

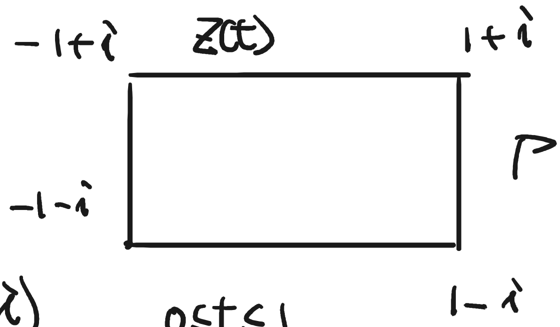
$$\overset{\text{case1}}{=} \left| \int_{\Gamma_i} g(z)\,dz \right|$$

M-L

$$\leq M \cdot \text{length}(\Gamma_i) < M \cdot \varepsilon/M = \varepsilon$$

$$\Rightarrow \int_\Gamma g(z)\,dz = 0 \qquad \#$$

Example



$$z(t) = \begin{cases} (1-t)(1+i) + t(-1+i), & 0 \leq t \leq 1 \\ (2-t)(-1+i) + (t-1)(-1-i), & 1 \leq t \leq 2 \\ (3-t)(-1-i) + (t-2)(1-i) & 2 \leq t \leq 3 \\ (4-t)(1-i) + (t-3)(1+i) & 3 \leq t \leq 4 \end{cases}$$

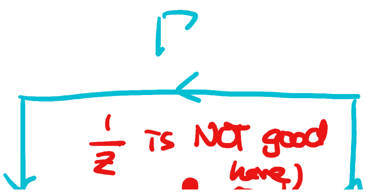
$$\int_\Gamma z^2\,dz = 0$$

$$\int_\Gamma z\,dz = 0$$

$$= \int_0^1 \Big[(1-t)(1+i) + t(-1+i)\Big]\Big[-(1+i) + (-1+i)\Big]\,dt$$

$$+ \int_1^2 \quad + \int_2^3 \quad + \int_3^4 \quad = 0$$

Remark



$$\int_\Gamma \frac{1}{z}\, dz \neq 0$$