Complex Analysis 3/7

Recall $C: z(t),\ a \le t \le b$

$$\int_C f(z)\,dz = \int_a^b f(z(t))\cdot z'(t)\,dt$$

Example (p. 48)

① $f(z) = x^2 + iy^2$, $\quad C: z(t) = t + it,\ 0 \le t \le 1$

$$\Rightarrow z'(t) = 1 + i$$

$$\int_C f(z)\,dz = \int_0^1 f(t+it)\cdot(1+i)\,dt$$

$$= \int_0^1 \overbrace{(t^2 + it^2)}^{(1+i)t^2}\cdot(1+i)\,dt$$

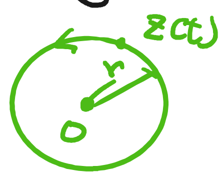
$$= (1+i)^2 \int_0^1 t^2\,dt = \frac{2i}{3}$$

*

② $$f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)}$$

$$= \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2}$$

C:



$$C: z(t) = \underbrace{r\cos t + i r\sin t}_{re^{it}},\quad 0 \le t \le 2\pi$$
$$r \ne 0$$

$$\int_C f(z)\,dz = \int_0^{2\pi} f(r\cos t + i r\sin t)\cdot(ire^{it})\,dt$$

$$= \int^{2\pi}\left(\frac{\cancel{r}\cos t}{} - i\,\frac{\cancel{r}\sin t}{}\right)\cdot(-\cancel{r}\sin t + i\cancel{r}\cos t)\,dt$$

$\frac{1}{re^{it}} = \frac{1}{\cancel{r}} e^{-it}$ $\qquad$ $i\cancel{r}e^{it}$

$$= \int_0^{2\pi} i \, dt = 2\pi i \qquad \#$$

③ $f(z) \equiv 1$, C: any piecewise C' curve

$$\Rightarrow \int_C f(z)\,dz = \int_a^b 1 \cdot z'(t)\,dt = z(b) - z(a) \qquad \#$$

<u>M-L inequality</u>

Recall: $\left(\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx \right)$

<u>Lemma 4.9</u>

Suppose $G: [a,b] \to \mathbb{C}$ is a continuous $\mathbb{C}$-valued function. Then

$$\left| \int_a^b G(t)\,dt \right| \le \int_a^b |G(t)|\,dt$$

<u>pf</u>

Suppose

$$\int_a^b G(t)\,dt = R \cdot e^{i\theta}, \quad R \ge 0$$

$$\Rightarrow R = e^{-i\theta} \int_a^b G(t)\,dt$$

$$= \int_a e^{\cdots} \cdot G(t)\, dt \qquad \circledast$$

Suppose $e^{-i\theta} \cdot G(t) = A(t) + iB(t)$, $\quad A, B : [a,b] \to \mathbb{R}$

Then by $\circledast$,

$$\overset{\mathbb{R}\ni}{R} = \overset{\in\mathbb{R}}{\int_a^b A(t)\,dt} + i \overset{\in\mathbb{R}}{\int_a^b B(t)\,dt} = \int_a^b A(t)\,dt$$

$$\Rightarrow \quad \int_a^b A(t)\,dt = R, \qquad \int_a^b B(t)\,dt = 0$$

$\Rightarrow$

$$R = \left| \int_a^b G(t)\,dt \right| = \int_a^b \mathrm{Re}\left(e^{-i\theta} \cdot G(t)\right) dt$$

$\mathrm{Re}(z) \le |\mathrm{Re}(z)| \le |z|$

$$\le \int_a^b \left| e^{-i\theta} G(t) \right| dt = \int_a^b |G(t)|\, dt$$

$\#$

<u>Recall</u>

Let C be a piecewise C^1 curve given by $z(t)$, $a \le t \le b$

Then the <u>length</u> of C is

$$L = \int_a^b |z'(t)|\, dt \quad \left(\int_a^b \sqrt{dx^2 + dy^2} \right)$$

$$= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

<u>Thm 4.10</u> (M-L formula)

Suppose (i) C is a piecewise C^1 curve of length L

(ii) f is continuous C

(iii) $|f(z)| \leq \underset{\sim}{M}^{\in \mathbb{R}}$ on C

Then
$$\left|\int_C f(z)\,dz\right| \leq M \cdot L$$

pf

Suppose C is given by $z(t)$, $a \leq t \leq b$. Then

$$\left|\int_C f(z)\,dz\right| = \left|\int_a^b f(z(t))\, z'(t)\,dt\right|$$

Lemma 4.9

$$\leq \int_a^b \underbrace{|f(z(t))|}_{\leq M} \cdot |z'(t)|\,dt$$

$$\leq \int_a^b M \cdot |z'(t)|\,dt = M \cdot L \quad \#$$

Line integral version of fundamental thm of Calculus

(Recall: $\int_a^b f'(x)\,dx = f(b) - f(b)$)

Prop 4.12

i.e. $F'(z)$ exists in a nbd of C in $\mathbb{C}$

Suppose $F(z)$ is an analytic on a piecewise C^1 curve C : $z(t)$, $a \leq t \leq b$. Then

$$\int_C F'(z)\,dz = F(z(b)) - F(z(a))$$

Recall:

$$\frac{d}{dt}U(x(t),y(t)) = U_x \cdot x'(t) + U_y \cdot y'(t)$$

pf

Let $F(z) = U(z) + iV(z)$, $a \le t \le b$

$$\gamma(t) = F(z(t)) = U(x(t)+iy(t)) + iV(x(t)+iy(t))$$

$$\Rightarrow \gamma'(t) = \frac{d}{dt}U(x(t)+iy(t)) + i\frac{d}{dt}V(x(t)+iy(t))$$

chain rule for $\mathbb{R} \to \mathbb{R}^2 \to \mathbb{R}$

$$= U_x(z(t))\cdot x'(t) + \underbrace{U_y}_{-V_x}(z(t))\cdot y'(t) + i\left(V_x(z(t))\cdot x'(t) + \underbrace{V_y}_{U_x}(z(t))\cdot y'(t)\right)$$

C-R: $U_x = V_y$, $U_y = -V_x$

Note: $F' = U_x + iV_x$

$$= U_x(z(t))\cdot x'(t) - V_x(z(t))\cdot y'(t) + iV_x(z(t))\cdot x'(t) + iU_x(z(t))\cdot y'(t)$$

$$= \left(U_x(z(t)) + iV_x(z(t))\right)\cdot\left(x'(t) + iy'(t)\right)$$

$$= F'(z(t))\cdot z'(t) \quad \left(= \frac{d}{dt}\underbrace{F(z(t))}_{\gamma(t)}\right)$$

$$\Rightarrow \int_C F'(z)\,dz = \int_a^b F'(z(t))\cdot z'(t)\,dt$$

if $\gamma = \gamma_1(t) + i\gamma_2(t)$

$$= \int_a^b \gamma'(t)\,dt = \gamma(b) - \gamma(b)$$

then $= \int_a^b \gamma_1'(t)\,dt + i\int_a^b \gamma_2'(t)\,dt$

$F(z(b)) - F(z(a))$ $= \gamma_1(b) - \gamma_1(a)$ ✓

$$= F(z(b)) - F(z(a))$$

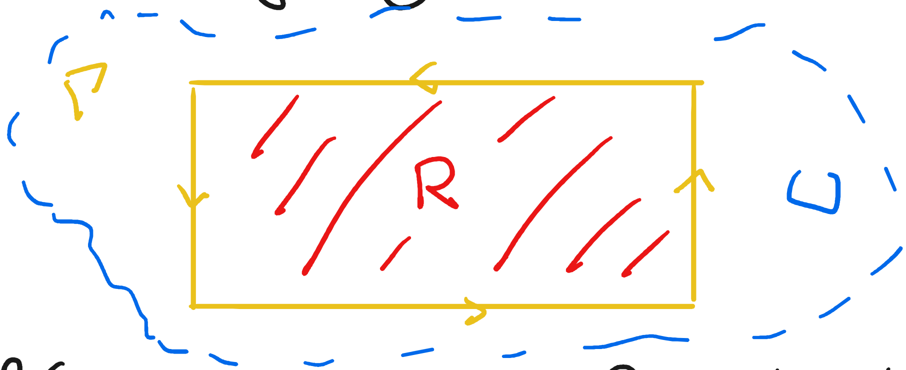
$+ i(\gamma_2(b) - \gamma_2(a))$

$= \gamma(b) - \gamma(a)$

Rectangle theorems

In this section, we assume Γ is the boundary of a rectangle R

$$R = \{x+iy \in \mathbb{C} \mid a \le x \le b,\ c \le y \le d\}$$



Goal (Thm 4.14, Thm 6.1, Rectangle Thm)

Suppose f is analytic on $U \underset{\text{open}}{\subseteq} \mathbb{C}$ and $R \subseteq U$

Then $$\int_\Gamma f(z)\,dz = 0$$

Lemma (p.52. a special case)

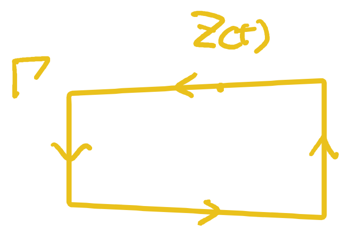
If f is a linear function, then

$$\int_\Gamma f(z)\,dz = 0$$

pf

$\alpha, \beta \in \mathbb{C}$

Let $f(z) = \alpha + \beta z$,



Γ be given $z(t)$, $a \le t \le b$, and

Γ be given $z(t)$, $a \le t \le b$, and

$$F(z) = \alpha z + \frac{\beta}{2} z^2$$

$$\Rightarrow \quad F'(z) = f(z) \quad \text{and} \quad z(a) = z(b)$$

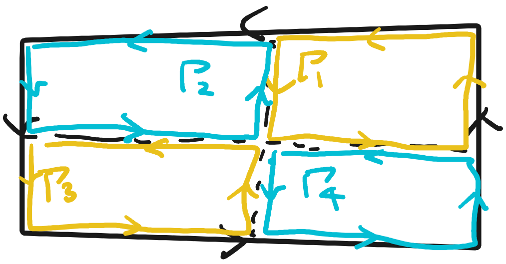
$$\overset{\text{Prop 4.12}}{\Longrightarrow} \quad \int_\Gamma f(z)\,dz = \int_\Gamma F'(z)\,dz$$

$$= F(z(b)) - F(z(a)) = 0 \quad \#$$

pf of Rectangle Thm

By Prop 4.7, we may assume Γ is counterclockwise

Idea: Γ



$$\int_\Gamma f(z)\,dz = \sum_{j=1}^{4} \int_{\Gamma_j} f(z)\,dz$$