Complex Analysis 3/3

Recall

- power series: $\sum_{n=0}^{\infty} c_n z^n$
- radius of convergence $= R = \dfrac{1}{\lim_{n\to\infty} |c_n|^{1/n}}$
- $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges and is differentiable in $|z| < R$

Example

$$\frac{1}{1-z} = \sum_{n=0}^{1} z^n \qquad \forall z \in D(0;R)$$

$\{z \in \mathbb{C} : |z-0| < R\} = D(0;R)$

$R = 1$

Remark (operations of power series, see HW3)

Suppose $\sum a_n z^n$ and $\sum b_n z^n$ have radii of convergence R_1 and R_2 respectively.

Then for $|z| < \min\{R_1, R_2\}$, the power series

$$\sum_{n=0}^{\infty} (a_n + b_n) z^n$$

and

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) z^n$$

converge and

converge, and

$$\sum_{n=0}^{\infty}(a_n+b_n)z^n = \sum_{n=0}^{\infty}a_n z^n + \sum_{n=0}^{\infty}b_n z^n$$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a_k b_{n-k}\right)z^n = \left(\sum_{n=0}^{\infty}a_n z^n\right)\cdot\left(\sum_{m=0}^{\infty}b_m z^m\right)$$

$$(a_0+a_1z+a_2z^2+\cdots)(b_0+b_1z+b_2z^2+\cdots)$$

$$\text{"}=\text{"}\sum_{m,n=0}^{\infty}a_nz^n\cdot b_mz^m = \sum_{m,n=0}^{\infty}a_nb_mz^{n+m}$$

$$=\sum_{N=0}^{\infty}\sum_{n+m=N}a_nb_mz^N = \text{LHS}$$

Example

Recall: $\frac{1}{1-z} = \sum_{n=0}^{\infty}z^n \qquad |z|<1$

Method I (last week)

$$\Rightarrow \left(\frac{1}{1-z}\right)' = \sum_{n=0}^{\infty}nz^{n-1} = \sum_{n=0}^{\infty}(n+1)z^n$$

$$= \frac{-(-1)}{(1-z)^2} = \frac{1}{(1-z)^2} = \frac{1}{1-z}\cdot\frac{1}{1-z}$$

Method II

$$= \left(\sum_{n=0}^{\infty}z^n\right)\cdot\left(\sum_{n=0}^{\infty}z^n\right)$$

by Remark

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}1\cdot 1\right)z^n = \sum_{n=0}^{\infty}(n+1)z^n$$

Remark

Power series in z at $a\in\mathbb{C}$ are of the form

$$\sum^{\infty} a_n(z-a)^n$$

$\sum_{n=0} c_n (z-\alpha)$

One can use the previous results by the substitution $\omega = z - \alpha$

Ch4 Line integrals

Introduction

By Thm 2.9, if $\sum c_n z^n$ converges $\forall z \in \mathbb{C}$, then $f(z) = \sum c_n z^n$ is an entire function

A big goal of Ch4-5: if $f(z)$ is entire then $\exists c_n \in \mathbb{C}$ s.t. $f(z) = \sum c_n z^n \quad \forall z \in \mathbb{C}$

($\Rightarrow$ f is ∞ differentiable)

Main tool: line integrals and

★★ Cauchy integral formula (Thm 5.3)

Line integrals

Def 4.1

Let $f: [a,b] \longrightarrow \mathbb{C}$ ($[a,b] \subseteq \mathbb{R}$) be continuous. Suppose

$$f(t) = u(t) + i\, v(t) \qquad t \in [a,b]$$

where $u, v : [a,b] \to \mathbb{R}$ Define

$$\int_a^b f(t)\,dt := \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

e.g.

$$\int_0^1 t + it^2\,dt = \int_0^1 t\,dt + i\int_0^1 t^2\,dt$$
$$= \frac{1}{2} + i\cdot\frac{1}{3}$$

Recall

A function $\gamma : [a,b] \to \mathbb{R}$ is C^1 if γ is continuous on $[a,b]$, differentiable on (a,b), and γ' is continuous on (a,b).

Def 4.2

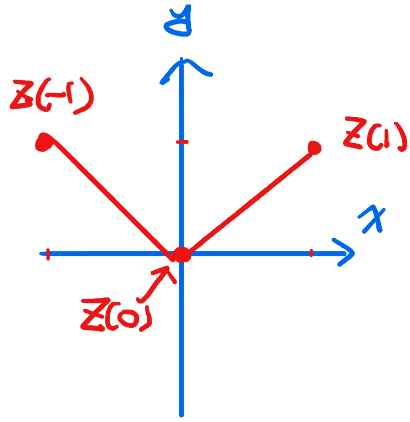
Let $z(t) = x(t) + iy(t)$, $a \le t \le b$, be a curve on $\mathbb{C}$. The curve is called piecewise C^1 if $x(t)$ and $y(t)$ are continuous on $[a,b]$ and

$\exists\ a < t_1 < t_2 < \cdots < t_n < b$

s.t. the restrictions of $x(t)$, $y(t)$ to intervals $[a, t_1]$, $[t_1, t_2]$, $\ldots$, $[t_{n-1}, b]$ are C^1.

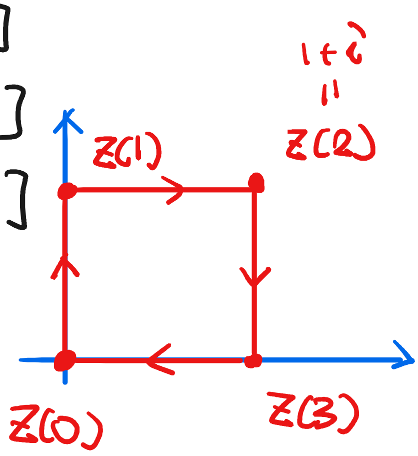
Example

① C_1 : $$z(t) = \begin{cases} t + it, & t \in [0,1] \\ t - it, & t \in [-1, 0] \end{cases}$$



is a piecewise C^1 curve

② C_2 : $$z(t) = \begin{cases} it, & t \in [0,1] \\ (t-1) + i, & t \in [1, 2] \\ 1 + (3-t)i, & t \in [2, 3] \\ 4 - t & t \in [3,4] \end{cases}$$



is a piecewise C^1 curve

<u>Recall</u> (change of variables for integration)

Suppose $\lambda : [a,b] \to \mathbb{R}$ is C^1, $f : \mathbb{R} \to \mathbb{R}$ is continuous. Then

$$\int_a^b f(\lambda(x))\, \underbrace{\lambda'(x)\, dx}_{\text{“}d\lambda\text{”}} = \int_{\lambda(a)}^{\lambda(b)} f(u)\, du$$

<u>Def 4.3</u>

Let C be a piecewise C^1 curve given by $z(t) = x(t) + iy(t)$, $a \le t \le b$.

Suppose $f : C \to \mathbb{C}$ is continuous

Then the (complex) integral of f

along C is

complex product

$$\int_C f(z)\,dz := \int_a^b f(z(t)) \cdot z'(t)\,dt$$

$$= \int_a^b f(z(t))(x'(t) + i y'(t))\,dt$$

Different!!

Remark (line integral in Calculus)

$$\int_C F(\gamma) \cdot d\gamma = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\,dt$$

"work in physics"   inner product!!

Example

① $f(z) = z^2$, $C = C_1$ in previous example

$$\int_{C_1} f(z)dz = \int_{-1}^{1} (z(t))^2 z'(t)\,dz$$

$$= \int_{-1}^{0} (t-it)^2 \cdot (1-i)\,dt + \int_{0}^{+1} (t+it)^2 \cdot (1+i)\,dt$$

$= (-2-2i)t^2$

$$= \int_{-1}^{0} (t^2 - 2it^2 - t^2)(1-i)\,dt$$

$= (2i-2)t^2$

$$+ \int_0^1 (t^2 + 2it^2 - t^2)(1+i)\,dt$$

$$= \frac{1}{3}(-2-2i) + \frac{1}{3}(2i-2) = -\frac{4}{3}$$ #

② $f(z) = z^2$, $C = C_2$

$$\int_{C_2} f(z)dt = \int_0^4 (z(t))^2 \cdot z'(t)\, dt$$
$$= \int_0^1 (it)^2 \cdot i\, dt + \int_1^2 (t-1+i)^2\, dt$$
$$+ \int_2^3 (1+(3-t)i)^2 (-i)\, dt + \int_3^4 (4-t)^2 (-1)\, dt$$
$$= 0 \qquad \#$$

<u>Remark for ②</u>

We will show that if $z(a) = z(b)$ and f is entire then $\int_C f(z)\,dz = 0$ (<u>Thm 4.16</u> = closed curve thm)

<u>Prop</u> (Change of variables)

Suppose $\lambda : [c,d] \to [a,b]$ is C^1 s.t. $\lambda(c) = a$, $\lambda(d) = b$. Then

$$\int_C f(z)dz = \int_a^b f(z(t))\, z'(t)\, dt$$
$$= \int_c^d f(z(\lambda(s))) \cdot z'(\lambda(s)) \cdot \lambda'(s)\, ds$$

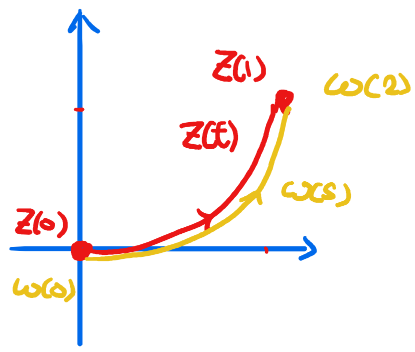
$t = \lambda(s) \Rightarrow dt = \lambda'(s)\, ds$

<u>Remark</u> (See <u>Prop 4.5</u>)

The above proposition $\Rightarrow$
if $z(t)$, $\omega(s)$ are 2 parametrizations of the same curve C with the same orientation, then

$$\int_a^b f(z(t))\, z'(t)\, dt = \int_C f(z)\, dz$$

$$= \int_c^d f(w(s)) \cdot w'(s)\, ds$$



e.g.

$$z(t) = t + it^2, \quad t \in [0,1]$$

$$w(s) = \frac{s}{2} + i\frac{s^2}{4}, \quad s \in [0,2], \quad f(z) = z$$

$$\int_C f(z)\, dz \overset{z(t)}{=} \int_0^1 (t + it^2) \cdot (1 + 2it)\, dt$$

$$= \int_0^1 (t - 2t^3) + i(t^2 + 2t^2)\, dt$$

$$= \frac{1}{2} - \frac{2}{4} + i = i$$

$$\overset{w(s)}{=} \int_0^2 \left(\frac{s}{2} + i\frac{s^2}{4}\right) \cdot \left(\frac{1}{2} + \frac{i}{4} \cdot 2s\right) ds$$

$$= \int_0^2 \left(\frac{s}{4} - \frac{s^3}{8}\right) + i\left(\frac{s^2}{8} + \frac{s^2}{4}\right) ds$$

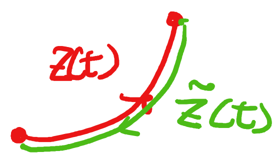
$$= \frac{4}{8} - \frac{2^4}{8 \cdot 4} + i\left(\frac{2^3}{8}\right) = i$$

Def 4.6

Suppose C is given by $z(t)$, $a \le t \le b$.

Then the curve $-C$ is defined by

$z(b + a - t)$, $a \le t \le b$.



e.g. $C: z(t) = t + it^2$, $t \in [0,1]$

$-C: \quad \tilde{z}(t) = z(1+0-t) = (1-t) + i(1-t)^2$

$t \in [0,1]$

<u>Prop 4.7</u>

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$$

pf

$$\int_{-C} f(z)\,dz = \int_a^b f(z(b+a-t)) \frac{d}{dt}\big(z(b+a-t)\big)\,dt$$

$$= \int_a^b f(z(b+a-t))\, z'(b+a-t) \cdot (-1)\,dt$$

$s = b+a-t$
$ds = -dt$

$$= -\int_a^b f(z(s))\, z'(s)\,ds = -\int_C f(z)\,dz \quad \#$$

<u>Prop 4.8</u> $\quad \alpha, \beta \in \mathbb{C}$

$$\int_C (\alpha f(z) + \beta g(z))\,dz$$

$$= \alpha \int_C f(z)\,dz + \beta \int_C g(z)\,dz$$

pf: exer