Complex Analysis 2/24

Power series (good for constructing analytic functions)

A power series in z (center at $0 \in \mathbb{C}$) is a series of the form

$$\sum_{k=0}^{\infty} c_k z^k, \qquad c_k \in \mathbb{C}$$

We say $\sum c_k z^k$ converges (resp. diverges) for $z_0 \in \mathbb{C}$ if the sequence

$$\left\{ \sum_{k=0}^{n} c_k z^k \right\}_{n=0}^{\infty}$$

converges (resp. diverges).

The limit is denoted by $\sum_{k=0}^{\infty} c_k z_0^k$.

Recall:

For real-valued seq. $\{a_k\}_{k=0}^{\infty} \subseteq \mathbb{R}$, its limsup is

$$\overline{\lim_{n\to\infty}}\, a_n = \limsup_{n\to\infty} a_n$$

$$:= \lim_{n\to\infty} \underbrace{\sup\{a_k : k \geq n\}}_{}$$

$a_n, a_{n+1}, a_{n+2}, \ldots$

this part is non-increasing

$\Rightarrow \overline{\lim} =$ a number or $\pm\infty$

<u>Example</u>

$a_n = (-1)^n$

or $+1, -1, +1, -1, \ldots$

$-1, +1, -1, +1, \ldots$

$$\Rightarrow \overline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty} \underbrace{\sup\{a_k : k \geq n\}}_{= +1}$$

$$= +1$$

The following theorems can be proved by arguments in Calculus:

<u>Thm 2.8</u> (root test)

Suppose $L = \overline{\lim_{n\to\infty}} |C_n|^{1/n}$

① If $L = 0$, then $\sum C_n z^n$ converges for all $z \in \mathbb{C}$.

② If $L = \infty$, then $\sum C_n z^n$ converges for $z = 0$ only.

③ If

③ If $0 < L < \infty$, set $R = \frac{1}{L}$

Then $\sum c_n z^n$ converges for $|z| < R$ and diverges for $|z| > R$.

__Def__

The number
$$R = \begin{cases} \infty & \text{if } L = 0 \\ \frac{1}{L} & \text{if } 0 < L < \infty \\ 0 & \text{if } L = \infty \end{cases}$$
is called the radius of convergence of the power series $\sum c_n z^n$

__Prop__ (exer 13, Ch2. ⇒ ratio test)

Suppose $\{a_n\}$ is a seq of positive real numbers and
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$$
Then $\lim_{n\to\infty} a_n^{\frac{1}{n}} = L$.

__Example__

c_n

Let $\sum c_n z^n = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$.

$|c_{n+1}|$ $\quad \frac{1}{\ }$

$$\frac{|c_{n+1}|}{|c_n|} = \frac{\frac{1}{\cancel{(n+1)!}\ n+1}}{\frac{1}{\cancel{n!}}} = \frac{1}{n+1} \xrightarrow[n\to\infty]{} 0$$

$$\Rightarrow \lim_{n\to\infty} |c_n|^{1/n} = 0 \Rightarrow R = \infty$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{n!} z^n \text{ converges for all } z \in \mathbb{C}. \quad \#$$

Thm 2.9

Let R be the radius of convergence of $\sum c_n z^n$. Let

$$f(z) := \sum_{n=0}^{\infty} c_n z^n, \qquad |z| < R$$

Then ① $f'(z)$ exists,

② radius of convergence of

$$\sum n c_n z^{n-1}$$

is R

③ $$f'(z) = \sum_{n=1}^{\infty} n c_n z^{n-1}, \qquad |z| < R.$$

Cor 2.10

apply Thm 2.9

$f(z) = \sum_{n=0}^{\infty} c_n z^n$ is infinitely differentiable in $D(0;R) := \{z \in \mathbb{C} : |z| < R\}$

= domain of convergence

where R = radius of convergence

Cor 2.11

If $f(z) = \sum_{n=0}^{\infty} c_n z^n$ has a nonzero radius of convergence, then

$$c_n = \frac{f^{(n)}(0)}{n!} \qquad \forall\, n \in \mathbb{N}$$

(because by Thm 2.9

$$f'(z) = \sum_{n=1}^{\infty} n\, c_n\, z^{n-1} \qquad |z| < \overset{\neq 0}{R}$$

$$\Rightarrow f'(0) = \sum_{n=1}^{\infty} n\, c_n\, 0^{n-1} = 1 \cdot c_1$$

Similarly,

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n\cdot(n-1)\cdots(n-k+1)\, c_n\, z^{n-k}, \quad |z|<R$$

$$\Rightarrow \quad f^{(k)}(0) = k(k-1)\cdots 1 \cdot c_k$$

$$\Rightarrow \quad c_k = \frac{f^{(k)}(0)}{k!}$$

)

Example

Let $f(z) = \sum_{n=0}^{\infty} z^n$ $\left(\overline{\lim_{n\to\infty}}\, 1^{\frac{1}{n}} = 1 \Rightarrow R = \frac{1}{1} = 1 \right)$

$|z| < 1$

Note that

$$\left(\sum_{n=0}^{N} z^n\right)(1-z) = \sum_{n=0}^{N} z^n - \sum_{n=0}^{N} z^{n+1}$$

$$= (1 + \cancel{z} + \cancel{z^2} + \cdots + \cancel{z^N})$$

$-(\cancel{z}+\cancel{z}+\cdots+\cancel{z}+z\quad)$

$= 1 - z^{N+1}$

$z^{N+1} \to 0$

$\Uparrow$

$$\Rightarrow \sum_{n=0}^{N} z^n = \frac{1-z^{N+1}}{1-z} \quad \text{for } |z|<1$$

$$\longrightarrow \frac{1}{1-z} \quad \text{as } N \to \infty$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{for } |z|<1$$

is analytic in $D(0;1)$

and
$$f'(z) = \sum_{n=1}^{\infty} n z^{n-1} = \frac{1}{(1-z)^2} \quad \text{for } |z|<1$$

\#

Thm2.12 (Uniqueness of power series)

Let $f(z) = \sum_{n=0}^{\infty} C_n z^n$.

Suppose $\exists$ nonzero seq. $\{z_k\} \subseteq \mathbb{C}$ s.t.

① $\lim_{k\to\infty} z_k = 0$

② $f(z_k)$ converges and $=0$ $\forall k$.

Then $C_n = 0$ $\forall n$, i.e. $f(z) \equiv 0$

pf

1° Since $f(z)$ converges at $z_k \neq 0$,

the radius of convergence of $f(z) = R > 0$ $\geq |z_k|$

Thm 2.9

$\Rightarrow f$ is differentiable ...

$\Rightarrow f$ is differentiable at 0

$\Rightarrow f$ is continuous at 0

$$\Rightarrow f(0) = f(\lim_{k\to\infty} z_k) = \lim_{k\to\infty} \overset{\overset{0}{\text{// by ②}}}{f(z_k)}$$

$$\boxed{\overset{\shortparallel}{C_0} = 0}$$

2° Note that

$$\frac{f(z_k)}{z_k} = \frac{\sum_{n=1}^{\infty} C_n z_k^{\cancel{n}\, n-1}}{\cancel{z_k}} = \boxed{\sum_{n=1}^{\infty} C_n z_k^{n-1}}$$

also converges, $= 0$ $\qquad \overset{!!}{g(z)}$

$\Rightarrow$ radius of convergence > 0

$\Rightarrow g(z)$ is also continuous at 0

$$\Rightarrow C_1 = g(0) = g(\lim_{k\to\infty} z_k)$$

$$= \lim_{k\to\infty} g(z_k) \overset{= f(z_k)/z_k = 0}{=} 0$$

3° Similarly, if $c_j = 0$ $\forall\ 0 \le j < n$, then

$$C_n = \lim_{k\to\infty} \frac{f(z_k)}{z_k^n} = 0$$

induction

$$\Rightarrow \quad C_n = 0 \quad \forall\ n$$

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Cor 2.14

If $\sum a_n z^n$ and $\sum b_n z^n$ converge and agree on a set S with an accumulation point at 0, then

$$a_n = b_n \quad \forall n$$

Recall

$p \in \mathbb{C}$ is an accumulation point of $S \subseteq \mathbb{C}$ if

$$D(p;r) \cap (S \setminus \{p\}) \neq \phi \quad \forall r > 0$$

pf of Cor 2.14

Assumption $\Rightarrow$ $\exists$ nonzero seq. $\{z_k\} \subseteq S \subseteq \mathbb{C}$ s.t.

① $\lim_{k\to\infty} z_k = 0$

② $\sum_{n=0}^{\infty} a_n z_k^n = \sum_{n=0}^{\infty} b_n z_k^n \quad \forall k$

$$\Rightarrow \sum_{n=0}^{\infty} \underbrace{(a_n - b_n)}_{c_n \text{ in Thm 2.12}} z_k^n = 0 \quad \forall k$$

by Thm 2.12

$\Rightarrow \quad a_n - b_n = 0 \quad \forall n$

$\Rightarrow \quad a_n = b_n \quad \forall n$ #

Example (proofs are outlined in homework)

① $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \quad \forall z \in \mathbb{C}$

(Recall: $e^{x+iy} = e^x(\cos y + i \sin y)$)

② $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \quad \forall z \in \mathbb{C}$

③ $\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} \quad \forall z \in \mathbb{C}$