Complex Analysis 2/21

Summary of last lecture

correction

Let $f: U \xrightarrow{\text{open } \subseteq \mathbb{C}} \mathbb{C}$, $u, v: U \to \mathbb{R}$, $f = u + iv$

- If f is differentiable at z, then

$$\boxed{f_y = i f_x} \iff \boxed{\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}}$$ Cauchy–Riemann equation

at z

$u, v: U^{\subseteq \mathbb{R}^2 \cong \mathbb{C}} \longrightarrow \mathbb{R}$

$u = u(x,y)$

$u_x(x,y) = \lim_{h \in \mathbb{R},\, h \to 0} \dfrac{u(x+h,y) - u(x,y)}{h}$

eg $u = x^2 y \Rightarrow u_x = 2x \cdot y$

- Suppose
  (i) f_x, f_y exist in a nbd of z
  (ii) f_x, f_y are continuous at z
  (iii) $\boxed{f_y = i f_x}$ at z ← correction

  Then f is differentiable at z.

- Example: Let

$$f(x+iy) := \begin{cases} \dfrac{xy(x+iy)}{x^2+y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

$$\Rightarrow f_x(0) = \lim_{\xi \to 0} \frac{\frac{\xi \cdot 0 (\xi + i \cdot 0)}{\xi^2 + 0^2} - f(0)}{\xi} = \lim_{\xi \to 0} \frac{0-0}{\xi} = 0$$

$$f_y(0) = \lim_{\eta \to 0} \frac{\frac{0 \cdot \eta (0 + i\eta)}{0^2 + \eta^2} - f(0)}{\eta} = \lim_{\eta \to 0} \frac{0-0}{\eta} = 0$$

$$\Rightarrow \boxed{f_y = i f_x \quad \text{at } z = 0}$$ (iii)

$$\Rightarrow f_x = \frac{(2xy + iy^2)(x^2+y^2) - (x^2y + ixy^2) \cdot 2x}{(x^2+y^2)^2}$$ ← if $(x,y) \neq (0,0)$

$$= \frac{2xy^3 + i(y^4 - x^2y^2)}{(x^2+y^2)^2} \not\to f_x(0) = 0$$

(f_x is NOT continuous at 0)

And $\lim\limits_{\substack{t\to 0\\ t\in\mathbb{R}}} \dfrac{f(0+t+i(0+t))-f(0)}{\underbrace{t+it}_{=h}} = \lim\limits_{t\to 0} \dfrac{\dfrac{\cancel{t}\,t(t\cancel{+}it)}{\cancel{t^2+t^2}\ 2}}{\cancel{t(1+i)}} = \dfrac{1}{2}$ ✳

$\neq \lim\limits_{\substack{\xi\to 0\\ \xi\in\mathbb{R}}} \dfrac{f(0+\xi)-f(0)}{\xi} = f_x(0) = 0$

$\Rightarrow$ $f'(z)$ does NOT exist at $z=0$

<u>Conclusion</u>:

f satisfies (i) and (iii)

f doesn't satisfy (ii)

f is NOT differentiable at $z=0$

- If f is (complex) differentiable at z_0, then

$Df(z_0)(z-z_0) = f'(z_0)\cdot(z-z_0)$

i.e. C-R

$$Df(z_0) = \begin{pmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{pmatrix} = \begin{pmatrix} a_0 & -b_0 \\ b_0 & a_0 \end{pmatrix}$$

$u_y(z_0) = -v_x(z_0)$, $v_y(z_0) = u_x(z_0)$

if $f'(z_0) = a_0 + i b_0$ $(= f_x(z_0) = u_x(z_0) + i\, v_x(z_0))$

<u>Analyticity</u>

<u>Def</u> 3.3

f is <u>analytic</u> at z if f is differentiable in a nbd of z

f is <u>analytic</u> on a set S if f

[illegible]

is differentiable in a nbd of S

f is an entire function if f is differentiable in whole $\mathbb{C}$

Note

Analyticity implies a lot of consequences. Following are 2 examples:

Prop 3.6

If $f = u + iv$ is analytic in a region (= open connected set in $\mathbb{C}$) U and u is constant, then f is constant in U.

"Not true in Advanced Calculus"
e.g. $\mathbb{R}^2 \to \mathbb{R}^2 : \binom{x}{y} \mapsto \binom{x+y}{1}$

pf

If $u = \text{constant}$, then

C-R

$$u_x = 0 = v_y$$

$$u_y = 0 = -v_x$$

U is connected

$\Rightarrow v = \text{constant} \Rightarrow f = \text{constant}$ #

Prop 3.7

If f is analytic in a region D and

If f is analytic in a region D and

$|f|$ is constant in D,

then f is constant in D.

pf

Case1: $|f| \equiv 0 \Rightarrow f \equiv 0$

Case2: $|f| \equiv c \neq 0 \Rightarrow \boxed{u^2 + v^2 \equiv c^2 \neq 0}$ $(*)$

$$\Rightarrow \frac{\partial (*)}{\partial x} : \quad 2u \cdot u_x + 2v \cdot v_x = \frac{\partial c^2}{\partial x} = 0$$

$$\frac{\partial (*)}{\partial y} : \quad 2u \cdot u_y + 2v \cdot v_y = \frac{\partial c^2}{\partial y} = 0$$

C-R $\Rightarrow$

$$\begin{cases} u u_x + v v_x = 0 = u u_x - v u_y & ① \\ u u_y + v v_y = 0 = u u_y + v u_x & ② \end{cases}$$

$\Rightarrow u \cdot ① + v \cdot ②$:

$$u^2 u_x - \cancel{uv} u_y + \cancel{vu} u_y + v^2 u_x = 0$$

$$(u^2 + v^2) \cdot u_x = c^2 \cdot u_x = 0$$

$c^2 \neq 0$

$$\Rightarrow u_x = 0 = v_y \quad \text{in } D$$

$u \cdot ② - v \cdot ①$:

$$u^2 u_y + \cancel{uv u_x} - \cancel{vu u_x} + v^2 u_y = 0$$

$$= (u^2+v^2)\, u_y = C\, u_y$$

$$\Rightarrow \quad u_y = 0 = -v_x \quad \text{in } D$$

So $f =$ constant in D #

Examples of entire functions

① A constant function is entire

② $f(z) = z$ is entire,

$f'(z) = 1$

Note
$f(x+iy) = x$
is NOT in $\underbrace{\mathbb{C}[z]}_{\{\text{poly in } \mathbb{C}\}}$

③ A polynomial $(a_i, z \in \mathbb{C})$

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n$$

is entire, and

$$p'(z) = a_1 + 2a_2 z + \cdots + (n-1)a_{n-1} z^{n-2} + n a_n z^{n-1}$$

④ Exponential map:

(Recall: for $x \in \mathbb{R}$, $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \cdots$)

For $z = x+iy \in \mathbb{C}$, we define

$$e^z = e^{x+iy} := e^x(\cos y + i \sin y)$$

Lemma

$f(z) = e^z$ is entire

pf

Write $f = u + iv$,

$$u = e^x \cos y, \quad v = e^x \sin y$$

Since

$$f_x = u_x + i v_x = \frac{\partial(e^x \cos y)}{\partial x} + i \cdot \frac{\partial(e^x \sin y)}{\partial x}$$

$$= \underline{e^x \cos y + i \cdot e^x \sin y}$$

$$f_y = u_y + i v_y = \frac{\partial(e^x \cos y)}{\partial y} + i \frac{\partial(e^x \sin y)}{\partial y}$$

$$= \underline{-e^x \sin y + i e^x \cos y}$$

exist, continuous in $\mathbb{C}$

and

$$f_y \overset{C\text{-}R}{=} i f_x$$

$$-e^x \sin y + i e^x \cos y = i(e^x \cos y + i e^x \sin y)$$

we conclude that

$$f(z) = e^z = e^x(\cos y + i \sin y)$$

is differentiable at all $z \in \mathbb{C}$ #

<u>Prop</u> For $z_1, z_2, z \in \mathbb{C}$, $x \in \mathbb{R}$,

- $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$
- $e^{-z} = \dfrac{1}{e^z}$
- $|e^z| = e^x$, $z = x+iy$
- $e^0 = 1$
- e^x = exportial defined in Calculus for $x \in \mathbb{R}$
- $(e^z)' = (e^z)_x = (e^x(\cos y + i\sin y))_x$
  $= e^x(\cos y + i \sin y) = e^z$

⑤ Trigonometric functions:

For $z \in \mathbb{C}$, we define

$$\sin z := \frac{1}{2i}(e^{iz} - e^{-iz})$$

$$\cos z := \frac{1}{2}(e^{iz} + e^{-iz})$$

which are entire functions