Complex Analysis 2/17

Recall

- Let $f : U \xrightarrow{} \mathbb{C}$, $z \in U$. ($U \subseteq \mathbb{C}$ open)

$$f'(z) = \lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z+h) - f(z)}{h}$$

- If f, g are differentiable at z, then so are $f+g$ and $f \cdot g$.
  And
$$(f+g)'(z) = f'(z) + g'(z)$$
$$(f \cdot g)'(z) = f'(z)\, g(z) + f(z) \cdot g'(z)$$

Example

① $f(z) = c =$ constant

$\Rightarrow f'(z) = 0 \quad \forall z \in \mathbb{C}$

② $f(z) = z \Rightarrow f'(z) = 1$

③ $f(z) = a z^n \Rightarrow f'(z) = n a z^{n-1}$

pf (induction)

- $n = 0, 1$: ① and ②

$$(a z)' = \underbrace{(a)'}_{=0} z + a \boxed{(z)'}^{=1} = a$$

$$(a z^{n+1})' = (a z^n \cdot z)' = (a z^n)' \cdot z + a z^n (z)'$$

induction hypothesis

$$= (n a z^{n-1}) \cdot z + a z^n$$

$$= (n+1) a z^n$$

#

④ $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$

is differentiable at any $z \in \mathbb{C}$

and

$$(a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0)'$$

$$= n a_n z^{n-1} + (n-1) a_{n-1} z^{n-2} + \cdots + a_1$$

⑤ $f(z) = f(x+iy) = \underline{x}$ is $\underline{\text{NOT}}$

(complex) differentiable at $z=0$ because

$$\lim_{\substack{h = \varepsilon \to 0 \\ \varepsilon \in \mathbb{R}}} \frac{\overset{=\varepsilon}{f(0+\varepsilon)} - \overset{=0}{f(0)}}{\varepsilon} = 1$$

$\cancel{\#}$

$$\lim \frac{\overset{=0}{f(0+i\delta)} - \overset{=0}{f(0)}}{} = 0$$

$h = i\delta \to 0$ $\quad i\delta$
$\delta \in \mathbb{R}$

\#

Prop 3.1 (Cauchy-Riemann eq.)

Suppose $u, v : U \to \mathbb{R}$ s.t. $f = u + iv$

If f is differentiable at z $\quad x+iy$,

then $f_x = u_x + i v_x$,

$f_y = u_y + i v_y$

exist and satisfy the

Cauchy-Riemann equation

$$f_y = i f_x$$

or equivalently

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

pf

Since $\boxed{\lim \dfrac{f(z+h) - f(z)}{\ldots}} = f'(z)$

$$\lim_{\substack{h\to 0,\\ h\in\mathbb{C}}} \frac{\quad}{h}$$

exists, we have

$$① \lim_{\substack{h = \xi \to 0\\ \xi\in\mathbb{R}}} \frac{f((x+\xi)+iy) - f(x+iy)}{\xi}$$

$$= f'(z) = \boxed{f_x(x+iy)}$$

$\because f'(z)$ exists

$$② \lim_{\substack{h = i\eta \to 0\\ \eta\in\mathbb{R}}} \frac{f(x+i(y+\eta)) - f(x+iy)}{i\eta}$$

$$= f'(z) = \boxed{\frac{1}{i} f_y(x+iy)}$$

So $\boxed{f_y = i f_x}$

$$\underbrace{f_y}_{=\, u_y + i v_y} \qquad \underbrace{i f_x}_{=\, i(u_x + i v_x) = i u_x - v_x}$$

$$\boxed{\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}}$$

#

Example

① $f(x+iy) = \underbrace{x}_{u} + i \cdot \underbrace{0}_{v}$

$\Rightarrow u_x = 1 \neq v_y = 0$

$\Rightarrow$ NOT differentiable. #

② Show that

$$f(x+iy) = \underline{x^2+y^3} + i\,\overset{v}{\boxed{xy}} + \underline{2xy^2}\ \ (u)$$

is NOT differentiable at $z = 1+i$

pf

$u(x,y) = x^2+y^3+2xy^2$

$v(x,y) = xy$

$\Rightarrow f = u + iv$

$u_x = 2x + 2y^2$ $\neq$ at $(1,1)$

$v_y = x$

$\Rightarrow f$ is NOT differentiable

$\Rightarrow f$ is ... differentiable

at $z = 1+i$ #

<u>Prop 3.2</u> (partial converse of <u>Prop 3.1</u>)

Suppose f_x and f_y exist in a neighborhood U of z.

If f_x and f_y are continuous at z and

更正

$$\boxed{f_y = i f_x} \quad \text{at } z,$$

then f is differentiable at z

<u>pf</u>

<u>Recall</u>: MVT

If $g: \mathbb{R} \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then $\exists\ c \in (a,b)$ s.t.

$$g'(c) = \frac{g(b)-g(a)}{b-a}$$

Let $f = u+iv$ We will estimate

$$\frac{f(z+h)-f(z)}{h}$$

$$= \underbrace{\frac{1}{h}\big(u(z+h)-u(z)\big)}_{①} + \underbrace{\frac{i}{h}\big(v(z+h)-v(z)\big)}_{②}$$

By MVT,

① $$\frac{u(z+h)-u(z)}{h} = \frac{u(x+\xi,\, y+\eta)-u(x,y)}{\xi+i\eta}$$

$$= \eta\cdot\frac{\boxed{u(x+\xi,\, y+\eta)-u(x+\xi,\, y)}}{(\xi+i\eta)\cdot\eta} + \frac{u(x+\xi,\, y)-u(x,y)}{\xi+i\eta}$$

$$\overset{\text{MVT}}{=} \frac{\eta}{\xi+i\eta}\,\underset{\text{// MVT}}{\boxed{u_y(x+\xi,\, y+\theta_1\eta)}}$$

$$+ \frac{\xi}{\xi+i\eta}\, u_x(x+\theta_2\cdot\xi,\, y)$$

for some $0<\theta_1, \theta_2<1$

Similarly,

② $$\frac{v(z+h)-v(z)}{h} = \frac{\eta}{\xi+i\eta}\, v_y(x+\xi,\, y+\theta_3\eta)$$

$$+ \frac{\xi}{\xi+i\eta}\, v_x(x+\theta_4\xi,\, y)$$

for some $0<\theta_3, \theta_4<1$

Thus,

$$\frac{\eta}{\xi+i\eta} f_y'(z)$$

$$\| \text{ by C-R}$$

$$\left|\frac{f(z+h)-f(z)}{h} - f_x(z)\right|$$

$$= \left|\frac{f(z+h)-f(z)}{h} - \left(\frac{\xi}{\xi+i\eta} f_x(z) + \boxed{\frac{i\eta}{\xi+i\eta} f_x(z)}\right)\right|$$

$$\text{①}+\text{②} = \left| \frac{\xi}{\xi+i\eta}\left(u_x(x+\theta_2\xi, \eta) - u_x(x,y) + i\left(v_x(x+\theta_4\xi, y) - v_x(x,y)\right)\right) + \frac{\eta}{\xi+i\eta}\left(u_y(x+\xi, y+\theta_1\eta) - u_y(x,y) + i\left(v_y(x+\xi, y+\theta_3\eta) - v_y(x,y)\right)\right)\right|$$

$$\leq |u_x(x+\theta_2\xi, \eta) - u_x(x,y)| + |v_x(x+\theta_4\xi, y) - v_x(x,y)| + |u_y(x+\xi, y+\theta_1\eta) - u_y(x,y)| + |v_y(x+\xi, y+\theta_3\eta) - v_y(x,y)|$$

$$\longrightarrow 0 \quad \text{as} \quad h = \xi + i\eta \to 0$$

$\because$ f_x, f_y are continuous at z

#

<u>Remark</u>

The assumptions in <u>Prop 3.2</u> are necessary:

Let $f(z) = f(x+iy) = \begin{cases} \dfrac{xy(x+iy)}{x^2+y^2} & z \neq 0 \end{cases}$

$\sim \quad 0 \qquad z=0$

$\Rightarrow$ $i\, f_x(0,0) = 0 = f_y(0,0)$

but $f'(0)$ does NOT exist

(f doesn't satisfy the assumptions of Prop3.2)

Complex differentiability v.s. differentiability

Let $f : U \overset{\subseteq \mathbb{C} \cong \mathbb{R}^2}{} \longrightarrow \mathbb{C} \cong \mathbb{R}^2$

Recall

① f is differentiable (as real vector-valued function) at $z_0 = (x_0, y_0) \leftrightsquigarrow x_0 + iy_0$

if $\exists$ linear map

$$Df(z_0) : \mathbb{R}^2 \to \mathbb{R}^2$$

($\|z\| = |z|$)

s.t.

$$\lim_{\substack{z \to z_0 \\ z \in \mathbb{R}^2}} \frac{\|f(z) - f(z_0) - Df(z_0)(z - z_0)\|}{\|z - z_0\|} = 0$$

② For $f = (u, v) = u + iv : \underset{\text{open}}{U} \to \mathbb{R}^2$, ($U \subseteq \mathbb{R}^2$)

if $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist and are continuous on U,

then f is differentiable on U as a real vector-valued function is a real vector-valued function

Prop

If $f = u + iv : \underset{z_0 \in}{U} \subseteq \mathbb{C} \cong \mathbb{R}^2 \longrightarrow \mathbb{C} \cong \mathbb{R}^2$ is complex differentiable at z_0,

then f is differentiable at z_0 as a real vector-valued function and

$$Df(z_0)(z-z_0) = f'(z_0)\cdot(z-z_0)$$

$$\begin{pmatrix} c & -d \\ d & c \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = (\underset{= u_x}{c} + i\underset{= v_x}{d}) \cdot (a+ib) = \begin{pmatrix} ac-bd \\ +i(ad+bc) \end{pmatrix}$$

$$\text{"}\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}\text{"} \qquad u_y = -v_x, \quad v_y = \text{"}u_x\text{"}$$

$$(a,b) \mapsto (ac-bd,\ ad+bc)$$

Example

$$f(z) = \bar{z} \quad \longleftrightarrow \quad f(x,y) = (x,-y)$$

NOT complex differentiable

but differentiable as real-vector function

Convention

In this course,

differentiable = complex differentiable

unless otherwise stated.