Complex Analysis 2/14

Ch1 Complex numbers

Complex numbers were from

$$\underbrace{x^2+1}_{\geq 1 > 0 \text{ if } x \in \mathbb{R}} = 0$$

Add "imaginary number" $i = \sqrt{-1}$

Def 1.1 ⇝ Viewpoint from algebra, §1.1

The field of complex numbers $\mathbb{C}$ is the set $\mathbb{R}\times\mathbb{R} = \{(a,b) \mid a,b \in \mathbb{R}\}$

// notation

$a+bi = a+b\sqrt{-1}$

together with the operations

$$(a,b)+(c,d) = (a+c,\ b+d)$$

$$(a,b)\cdot(c,d) = (ac-bd,\ ad+bc)$$

e.g. $i \cdot i$

$$\underbrace{(0,1)\cdot(0,1)} = (0-1,\ 0+0)$$
$$= (-1,\ 0) \quad = \ -1$$

Prop

$(\mathbb{C}, +, \cdot)$ is a "field", i.e. $\forall x, y, z \in \mathbb{C}$,

- $x+(y+z) = (x+y)+z$, $\quad x\cdot(y\cdot z) = (x\cdot y)\cdot z$
- for $0 = (0,0)$, $1 = (1,0) \in \mathbb{C}$.

  $x + 0 = x$, $\quad x \cdot 1 = x$
- $x+y = y+x \qquad x\cdot y = y \cdot x$
- for $x = (a,b)$, $\quad x + (-a,-b) = 0$
- for $x = (a,b) \neq (0,0)$,

  $\exists \quad x^{-1} = \left(\dfrac{a}{a^2+b^2}, \dfrac{-b}{a^2+b^2}\right) \in \mathbb{C}$ s.t.

  $x \cdot x^{-1} = 1$
- $x\cdot(y+z) = x\cdot y + x \cdot z$

Remark

$(\mathbb{R}, +, \cdot) \hookrightarrow (\mathbb{C}, +, \cdot) : a \mapsto (a, 0)$

is an embedding of fields

Thm (will be proved in **Thm 5.12**, p66)

Every nonconstant polynomial equation

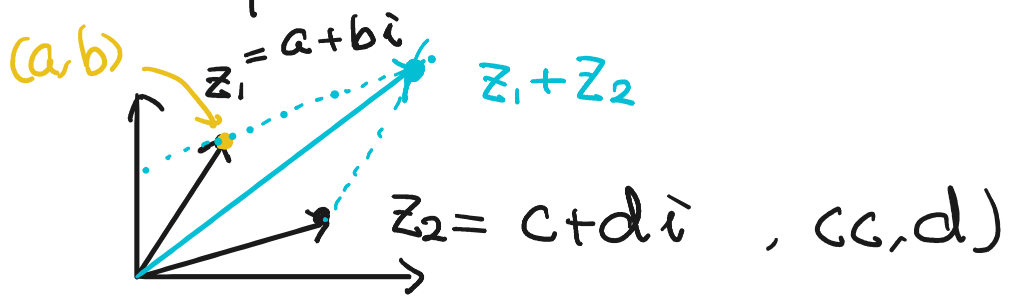
$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0, \quad a_i \in \mathbb{C}$$

has a solution in $\mathbb{C}$

Fundamental Thm of Algebra

Complex plane (viewpoint from linear alg or plane geometry, §1.2)

$(\mathbb{C}, +) \overset{\text{vector space}}{\cong} (\mathbb{R}^2, +)$ is 2-dim vector space over $\mathbb{R}$



Def

For $z = \underbrace{x+iy}_{=(x,y)} \in \mathbb{C}$,

$\operatorname{Re} z$ = real part of $z = x$

$\operatorname{Im} z$ = imaginary part of $z = y$

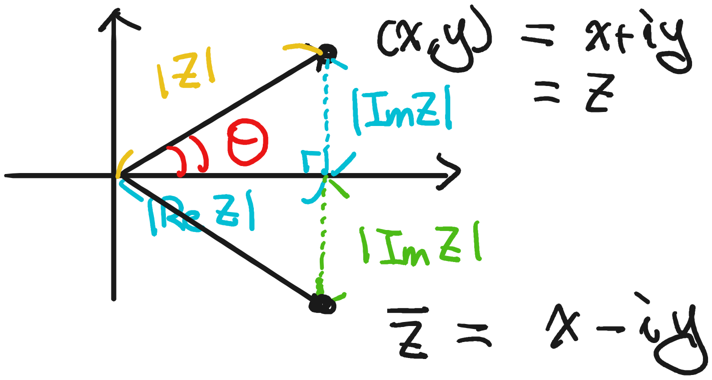
$\bar{z}$ = conjugate of z = $x - iy$

$|z|$ = absolute value of z

= modulus of z = $\sqrt{x^2+y^2}$

$\operatorname{Arg} z$ = argument of z

$= \theta$ s.t. $\begin{cases} \cos\theta = \dfrac{\operatorname{Re} z}{|z|} \\ \sin\theta = \dfrac{\operatorname{Im} z}{|z|} \end{cases}$



__Remark__

For $z_1, z_2 \in \mathbb{C}$, let $r_j = |z_j|$, $\theta_j = \operatorname{Arg} z_j$

Then

$$z_j = r_j(\cos\theta_j + i\sin\theta_j)$$

and

$$z_1 \cdot z_2 = r_1 r_2 (\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

$$z_1^{-1} = \frac{1}{r_1}(\cos(-\theta_1) + i\sin(-\theta_1))$$

assume $z_1 \neq 0$

$$= r_1^{-1}(\cos\theta_1 - i\sin\theta_1)$$

z_2 $r_2 (\cos(\theta_2 - \theta_1) + i\sin(\theta_2-\theta_1))$

$$\frac{z_2}{z_1} = \frac{r_2}{r_1}(\cos(\theta_2-\theta_1) + i\sin(\theta_2-\theta_1))$$

$$z_1^n = r_1^n(\cos(n\theta_1) + i\sin(n\theta_1)), \ \forall n \in \mathbb{Z}$$

Topological aspects of $\mathbb{C}$ §1.4

(viewpoint from advanced calculus)

$\mathbb{C}$ together with $d(z_1, z_2) = |z_1 - z_2|$

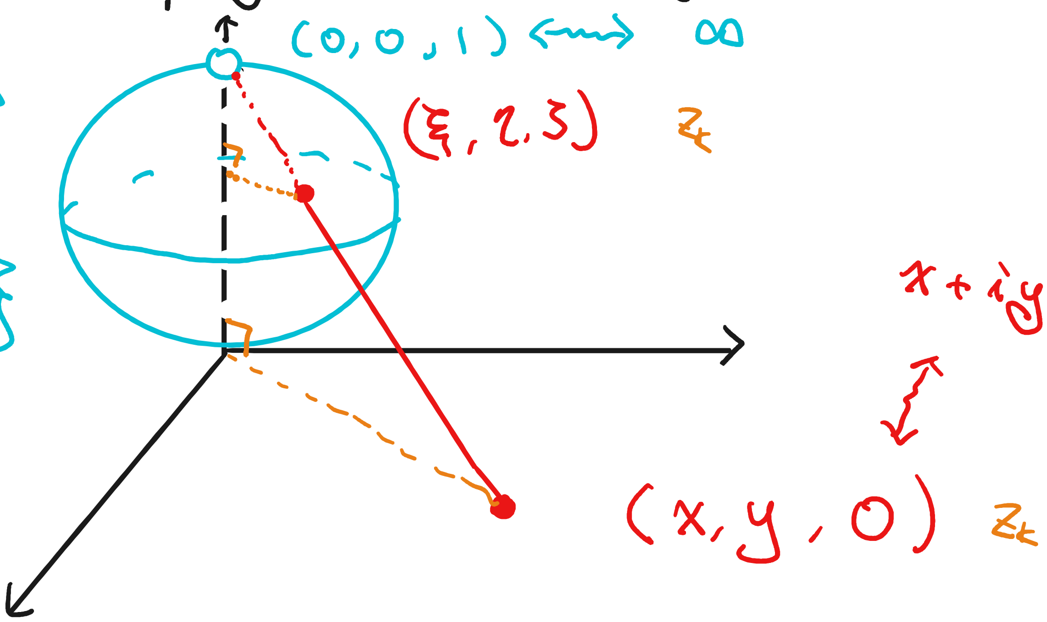
$$d(x_1+iy_1,\ x_2+iy_2) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$$

is a metric space

(In fact, $(\mathbb{C}, |\cdot|)$ is a normed vector space)

Stereographic projection (§1.5)

$\{(\xi, \eta, \zeta) \mid \xi^2 + \eta^2 + (\zeta - \frac{1}{2})^2 = (\frac{1}{2})^2\}$



$$\frac{x}{\xi} = \frac{y}{\eta} = \frac{1}{1-\zeta} \quad \text{(by similar triangles)}$$

$\Rightarrow$

- $x = \dfrac{\xi}{1-\zeta}$, $y = \dfrac{\eta}{1-\zeta}$

- $\xi = \dfrac{x}{x^2+y^2+1}$, $\eta = \dfrac{y}{x^2+y^2+1}$

  $\zeta = \dfrac{x^2+y^2}{x^2+y^2+1}$

Def

We say $\{z_k\} \to \infty$ if $|z_k| \to \infty$

$\lim_{z \to a} f(z) = \infty$ if $\lim_{z \to a} |f(z)| = \infty$

Remark

$\{z_k\} \to \infty$ on $\mathbb{C}$ $\Longleftrightarrow$ $\{z_k\} \to (0,0,1)$ on sphere

Ch 2-3 Analytic functions

Def

Let $f : U \longrightarrow \mathbb{C}$, $z \in U$. (U open $\subset \mathbb{C}$)

We say f is (complex) differentiable at z if

the limit

$$\lim \frac{f(z+h) - f(z)}{}$$

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \quad h$$

exists. In this case, the limit is denoted $f'(z)$

<u>Remark</u>

The limit is taken from all the possible directions in $\mathbb{C}$. So the condition "complex differentiable" is much stronger than "differentiable as functions of real variables"

Since the limit is of the same form as what we did in Calculus, we have

<u>Prop 2.5</u>

If f, g are differentiable at z, then so are

$$h_1 = f+g \quad , \quad h_2 = f \cdot g$$

and if $g(z) \neq 0$,

$$h_3 = \frac{f}{g}$$

In the respective cases,

$$h_1'(z) = f'(z) + g'(z)$$

$$h_2'(z) = f'(z)\cdot g(z) + f(z)\cdot g'(z)$$

$$h_3'(z) = \frac{f'(z)\cdot g(z) - f(z)\cdot g'(z)}{(g(z))^2}$$

pf: exer 6, Ch2

<u>Example</u>

① $f(z) = 2 \quad \forall z \in \mathbb{C}.$

$$\Rightarrow f'(z) = \lim_{h\to 0} \frac{\overbrace{f(z+h)}^{2=} - \overbrace{f(z)}^{=2}}{h} = 0$$

② $f(z) = z$

$$\Rightarrow f'(z) = \lim_{h\to 0} \frac{\overset{z+h}{\overset{\shortparallel}{f(z+h)}} - \overset{z}{\overset{\shortparallel}{f(z)}}}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

③ So

$$(2z)' = 0\cdot z + 2\cdot 1 = 2$$

$$(z+2)' = 1 + 0 = 1$$

$$(z^2)' = (z\cdot z)' = 1\cdot z + z\cdot 1 = 2z$$

<u>Prop</u> (exer 3, Ch2)

If f is (complex) differentiable at z,

g " $f(z)$,

then

$g \circ f$ " " z, and

$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z)$$

pf: exer