Complex Analysis — Homework 8

1. Evaluate
   (a) $\displaystyle\int_{|z|=1} \cot z \, dz$
   (b) $\displaystyle\int_{|z|=2} \frac{dz}{(z-4)(z^3-1)} \, dz$
   (c) $\displaystyle\int_{|z|=1} \sin \frac{1}{z} \, dz$
   (d) $\displaystyle\int_{|z|=2} z e^{3/z} \, dz$.

2. Prove that for any positive integer n, $\operatorname{Res}\left((1-e^{-z})^{-n}\,;\,0\right) = 1$.

3. Suppose that f is entire and that $f(z)$ is real if and only if z is real. Show that f can have at most one zero.

4. Find the number of zeros (counting multiplicities) of
   (a) $f_1(z) = 3e^z - z$ in $|z| \leq 1$.
   (b) $f_2(z) = \frac{1}{3}e^z - z$ in $|z| \leq 1$
   (c) $f_3(z) = z^4 - 5z + 1$ in $1 \leq |z| \leq 2$
   (d) $f_4(z) = z^6 - 5z^4 + 3z^2 - 1$ in $|z| \leq 1$.

5. Suppose f is analytic inside and on a regular closed curve γ and has no zeros on γ. Show that if m is a positive integer, then

$$\frac{1}{2\pi i} \int_\gamma z^m \frac{f'(z)}{f(z)} \, dz = \sum_k (z_k)^m$$

   where the sum is taken over all the zeros of f inside γ.

6. Recall that

$$\log(z^2 - 1) = \int_{\sqrt{2}}^{z} \frac{2\zeta}{\zeta^2 - 1} \, d\zeta$$

   is analytic in the plane minus the interval $(-\infty, 1]$, and thus so is

$$\sqrt{z^2 - 1} = \exp\left(\frac{1}{2}\log(z^2 - 1)\right).$$

   Show that the above $\sqrt{z^2-1}$ can be extended to a function which is analytic in the plane minus the interval $[-1, 1]$.