# Complex Analysis — Homework 7

1. Read the topological definitions of fundamental groups and simply connected spaces.
2. Let $D$ be an open connected set in $\mathbb{C}$. Show that, for any $z_0, z_1 \in D$, there exists a piecewise $C^1$ curve $\gamma : [0, 1] \to D$ such that $\gamma(0) = z_0$ and $\gamma(1) = z_1$.
3. A set $S$ is called *star-like* if there exists a point $\alpha \in S$ such that the line segment connecting $\alpha$ and $z$ is contained in $S$ for all $z \in S$. Show that a star-like region is simply connected. In particular, every convex region is simply connected.
4. Let $D = \{z \in \mathbb{C} : z \neq 0\}$ be the punctured plane. Show that the curve
$$\gamma : [0, 1] \to D,\ \gamma(t) = e^{2\pi t i},$$
is *not* homotopic to a constant curve.
5. Suppose that $f$ is analytic in $D = \{z \in C : |z| < 5, z \neq \pm 2\}$, and that $C, C_1, C_2$ are the circles
$$C = \{4e^{i\theta} : 0 \le \theta \le 2\pi\}, \quad C_1 = \{-2 + e^{i\theta} : 0 \le \theta \le 2\pi\}, \quad C_2 = \{2 + e^{i\theta} : 0 \le \theta \le 2\pi\}.$$
Show that
$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$
6. Define a function $f$ analytic in the plan minus the non-positive real axis and such that $f(x) = x^x$ on the positive axis. Find $f(i)$, $f(-i)$. Show that $f(\overline{z}) = \overline{f(z)}$ for all $z$.
7. Suppose $f(z) \to \infty$ as $z \to z_0$, where $z_0$ is an isolated singularity of $f$. Show that $f$ has a pole at $z_0$.
8. Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire one-to-one function. Show that $f(z) = az + b$ for some $a, b \in \mathbb{C}$.
9. Suppose $f$ is analytic in the punctured plane $z \neq 0$ and satisfies $|f(z)| \le \sqrt{|z|} + \dfrac{1}{\sqrt{|z|}}$. Prove $f$ is constant.
10. Classify the singularities of
    (a) $\dfrac{1}{z^4 + z^2}$
    (b) $\cot z$
    (c) $\csc z$
    (d) $\dfrac{\exp(1/z^2)}{z - 1}$.