Complex Analysis — Homework 6

1. Suppose that f is entire and that $|f(z)| \geq |z|^N$ for sufficiently large z. Show that f must be a polynomial of degree at least N.

2. Find the maximum and minimum moduli of $z^2 - z$ in the disc: $|z| \leq 1$.

3. Show that if f is analytic and nonconstant on a compact set in $\mathbb{C}$, then $\operatorname{Re} f$ and $\operatorname{Im} f$ assume their maxima and minima on the boundary.

4. Let $D = D(0;1)$ be the unit disc and $S^1 = \partial D$ be its boundary. Suppose f is nonconstant and analytic in D and continuous in its closure $\overline{D}$. Show that if $f(S^1) \subset S^1$, then $f(D) = D$.

5. Suppose f is entire and $|f| = 1$ on $|z| = 1$. Prove that there exists $c \in \mathbb{C}$ such that $f(z) = cz^n$ for all $z \in \mathbb{C}$.

6. Suppose that f is analytic in the annulus: $1 \leq |z| \leq 2$, that $|f| \leq 1$ for $|z| = 1$ and that $|f| \leq 4$ for $|z| = 2$. Prove $|f(z)| \leq |z|^2$ throughout the annulus.

7. (a) Suppose that f is analytic and bounded by 1 in the unit disc with $f(\alpha) \neq 0$ for some $|\alpha| < 1$. Show that there exists a function g , analytic and bounded by 1 in the unit disc, with $|g'(\alpha)| > |f'(\alpha)|$.

   (b) Find $\max_f |f'(\alpha)|$ where f ranges over the class of analytic functions bounded by 1 in the unit disc, and α is a fixed point with $|\alpha| < 1$.

8. Let
$$f(z) = \int_0^1 \frac{\sin zt}{t}\, dt.$$
   Show that

   (a) f is entire;

   (b) $f'(z) = \displaystyle\int_0^1 \cos zt\, dt$.

9. Given an entire function which is real on the real axis and imaginary on the imaginary axis, prove that it is an odd function, i.e., $f(z) = -f(-z)$.

10. Suppose f is analytic in $|z| < 1, \operatorname{Im} z > 0$, continuous on $|z| \leq 1, \operatorname{Im} z > 0$ and real on the semi-circle: $|z| = 1$, $\operatorname{Im} z > 0$. Show that if we set
$$g(z) = \begin{cases} f(z), & |z| \leq 1, \operatorname{Im} z > 0, \\ \overline{f(1/\overline{z})}, & |z| > 1, \operatorname{Im} z > 0, \end{cases}$$
    then g is analytic in the upper half plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.