problem 10 in h.w. 6：

Suppose f is analytic in the semi-disc: $|z|<1$, $\text{Im } z>0$, continuous on $|z|\le 1$, $\text{Im } z>0$, and real on the semi-circle $|z|=1$, $\text{Im } z>0$. Show that if we set

$$g(z)=\begin{cases} f(z), & |z|\le 1, \quad \text{Im } z>0, \\ \overline{f(1/\overline{z})}, & |z|>1, \quad \text{Im } z>0, \end{cases}$$

then g is analytic in the upper plane $\{\, z\in\mathbb{C} \mid \text{Im } z>0\}$.

1°

首先解釋為什麼題目要改成上面紅字那樣：

在原本的題目中，條件有 g is analytic on $|z|\le 1$, $\text{Im } z>0$ ，在一點 analytic 的定義是 在該點附近有個小範圍 differentiable ，也等同於在該點附近 analytic ，因此，如果是 analytic on $|z|\le 1$, $\text{Im } z>0$ ，其實是指 analytic on an open set containing $|z|\le 1$, $\text{Im } z>0$ ，如此一來，就變得只需要證明 analytic on $|z|>1$, $\text{Im } z>0$ ，將用不到 “g is real on the semi-circle ” 這個條件 事實上，這個條件是 g 能在 semi-circle 上的點連續的必要條件，上面題目敘述中的紅字連續那部分的意思應該要是“g 限制在 $|z|\le 1$, $\text{Im } z>1$ 時，是個連續函數” 才比較合理。

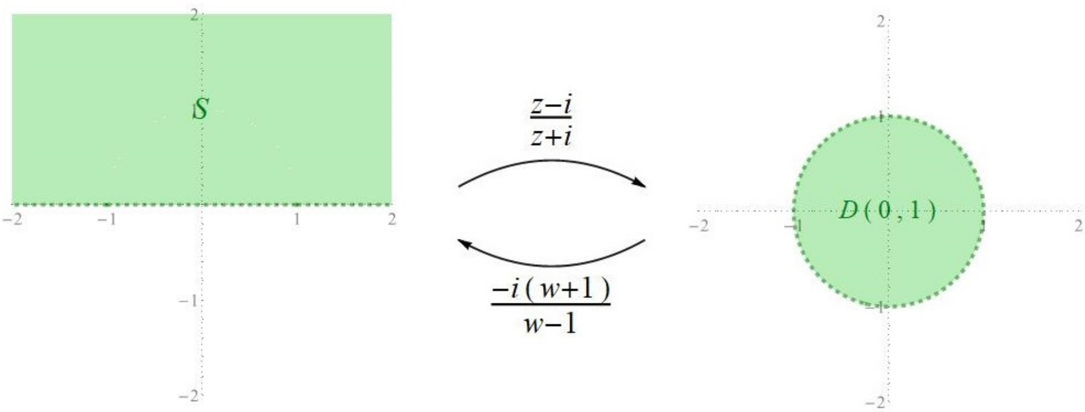
2°

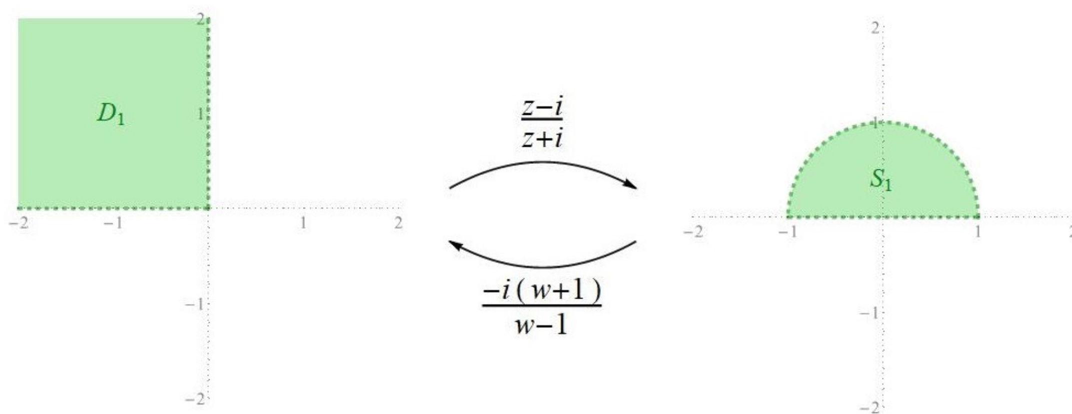
做這題的一個方法是將題目中給的半圓內區域，透過某種映射轉換成適合的區域，再用 Schwartz 反射定理 extend 到另一側，之後再用該映射的 inverse 轉換回來，最後說明轉換回來的函數即題目中定義的 g。

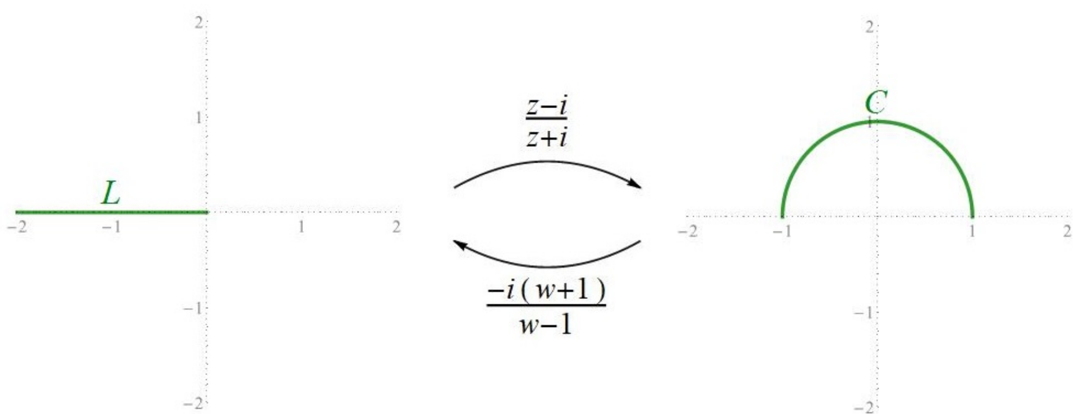
我們先用符號表示一些區域: Let

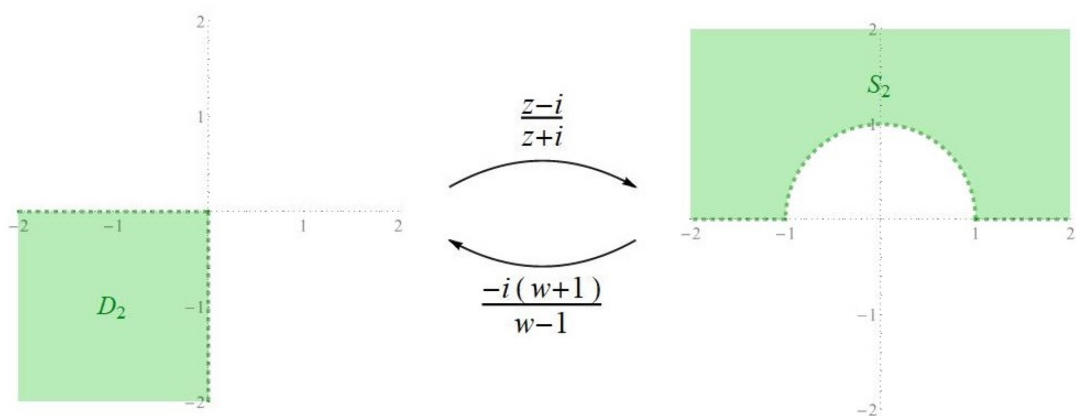
$S_1=\{\, z\in\mathbb{C} \mid |z|<1,\ \text{Im } z>0\}$, $C=\{\, z\in\mathbb{C} \mid |z|=1,\ \text{Im } z>0\}$, $S_2=\{\, z\in\mathbb{C} \mid |z|>1,\ \text{Im } z>0\}$,
$D_1=\{\, z\in\mathbb{C} \mid \text{Re } z<0,\ \text{Im } z>0\}$, $L=\{\, z\in\mathbb{C} \mid \text{Re } z<0,\ \text{Im } z=0\}$, $D_2=\{\, z\in\mathbb{C} \mid \text{Re } z<0,\ \text{Im } z<0\}$,
$S=S_1\cup D\cup S_2$, and $D=D_1\cup L\cup D_2$.

我們令 $\varphi(z)=\dfrac{z-i}{z+i}$ ，然後去證明 $\varphi^{-1}(w)=\dfrac{-i(w+1)}{w-1}$ ，說明如下一頁中幾張圖的對應關係、φ and φ^{-1} 在以下我們要用到的區域是 analytic 。 接著在 $D_1\cup L$ 上定義 $h(z)=f(\varphi(z))$ ，用 Schwartz reflection principle 將 h extend 到 D 上為 $\tilde{h}$ ，接著證明 $g(z)=\tilde{h}(\varphi^{-1}(z))$ ，便可說明 g 是 analytic 。


S
z−i
z+i
−i (w+1)
w−1
D (0 , 1)


D1
z−i
z+i
−i (w+1)
w−1
S1


L
z−i
z+i
−i (w+1)
w−1
C


D2
z−i
z+i
−i (w+1)
w−1
S2

首先，我們先說 φ 跟 $\varphi-1$ 在那些區域是解析函數，並且有如上圖的區域對應關係。

Since $\dfrac{1}{z+i}$ is analytic at every $z \in \mathbb{C}$, $z \neq -i$, the function $\varphi(z) = \dfrac{z-i}{z+i}$ is an analytic on $\mathbb{C} \setminus \{-i\}$

For $z \in \mathbb{C}$, we have $|z-i| < |z+i|$ if and only if $\operatorname{Im} z > 0$, therefore $\varphi(z) \in D(0,1)$ if and only if $z \in S$, and $\varphi(z) \in \mathbb{C} \setminus D(0,1)$ if and only if $\operatorname{Im} z < 0$, $z \neq -i$.

Solving $w = \varphi(z) = \dfrac{z-i}{z+i}$, we obtain $z = \dfrac{-i(w+1)}{w-1}$ for $w \neq 1$, therefore $\varphi^{-1}(w) = \dfrac{-i(w+1)}{w-1}$ defined on $\mathbb{C} \setminus \{1\}$.

Since $\varphi(z) = \dfrac{(z-i)(\bar{z}-i)}{|z+i|^2} = \dfrac{(|z|^2-1) - i\ 2\operatorname{Re} z}{|z+i|^2}$, we have $\operatorname{Im}(\varphi(z)) > 0$ if and only if $\operatorname{Re} z < 0$, and hence $\varphi(D_1) = S_1$, $\varphi(D_2) = S_2$.

接著我們在 $D_1 \cup L$ 上定義 $h = f \circ \varphi$ ，它在 D_1 上可解析，在 $D_1 \cup L$ 上連續，在 L 上取值為實數，因此由 Schwartz reflection principle ，它可以 extend 為一個解析函數

$$\tilde{h}(z) = \begin{cases} h(z), & \text{if} \quad z \in D_1 \cup L \\ \overline{h(\bar{z})}, & \text{if} \quad \bar{z} \in D_1 \end{cases}$$

定義在 D 上。

想說明 $g(z) = h(\varphi^{-1}(z))$ ，對於 $z \in S_1 \cup C$ 的部分，由 $h(z) = f(\varphi(z))$ 自然得到； 對於 $z \in S_2$ ，我們要算一下：由於 $\tilde{h}\left(\varphi^{-1}(z)\right) = \overline{h\left(\overline{\varphi^{-1}(z)}\right)} = \overline{f\left(\varphi\left(\overline{\varphi^{-1}(z)}\right)\right)}$ ，所以我們只要證明 $\varphi(\overline{\varphi^{-1}(z)}) = \dfrac{1}{\bar{z}}$ 即可。

Since

$$\varphi(\overline{\varphi^{-1}(z)}) = \varphi\left(\frac{i(\bar{z}+1)}{\bar{z}-1}\right) = \frac{\frac{i(\bar{z}+1)}{\bar{z}-1} - i}{\frac{i(\bar{z}+1)}{\bar{z}-1} + i} = \frac{i(\bar{z}+1) - i(\bar{z}-1)}{i(\bar{z}+1) + i(\bar{z}-1)} = \frac{1}{\bar{z}},$$

we obtain that

$$\tilde{h}(\varphi^{-1}(z)) = \overline{f(1/\bar{z})} = g(z)$$

for all $z \in S_2$.

$\Rightarrow g(z) = \tilde{h}(\varphi^{-1}(z))$ for all $z \in S$, and hence g is analytic on S. $\square$

這個方法不困難，主要就是要知道有一對一的解析函數能將某圓對應到某直線，圓內的區域送到半平面。 如果想瞭解更多相關的內容，可以看看書上的 section 13.2 或是搜尋 Möbius transformation

$3°$

這邊我們介紹另一個方法，用 Morera 定理來證明 g 是個 analytic function 。

我們先證明 g 在 S_2 裏頭可解析：

Let $z \in S_2$. Then for $w \in S_2$,

$$\frac{g(w)-g(z)}{w-z} = \frac{\overline{f(1/\bar{w})} - \overline{f(1/\bar{z})}}{w-z} = \overline{\left(\frac{f(1/\bar{w})-f(1/\bar{z})}{\bar{w}-\bar{z}}\right)}.$$

Since f is analytic on S_1 and $1/\bar{z} \in S_1$, we have

$$\frac{f(1/\bar{w})-f(1/\bar{z})}{\bar{w}-\bar{z}} = \frac{f(1/\bar{w})-f(1/\bar{z})}{(1/\bar{w})-(1/\bar{z})} \cdot \frac{-1}{\bar{w}\,\bar{z}}$$

tends to $f'(1/\bar{z}) \cdot \dfrac{1}{\bar{z}^2}$ as $w \to z$.

$\Rightarrow g'(z) = \overline{f'(1/\bar{z})}\,\dfrac{-1}{z^2}$, and hence g is analytic on S_2.

接下來我們將要用 Morera 定理，先證明 g 在整個 S 上連續，先取一個 S 中的長方形路徑，去證明 g 沿著這條路徑的積分是 0 。 這部分有幾種方式可以處理，我將上演習課時講的方式打下： (一些集合的符號沿用上面的)

$3°$-1

首先我們證明 g 在 C 上也是連續的 (這部分分別考慮從 S_1 及 S_2 中的點逼近):

For $z_0 \in C$, $z \in S_1$, $g|_{S_1 \cup C} = f$, by the continuity of f at z_0, tends to 0 as $z \to z_0$.

Let $T(z) = 1/\bar{z}$. Then for each $r > 0$ and each $\theta \in [0, 2\pi]$, $T(re^{i\theta}) = r^{-1}e^{i\theta}$, and therefore $T(z) = z$ for every $z \in C$, $T(S_2) = S_1$, and $T(S_1) = S_2$.

Since $g|_{S_2 \cup C} = f \circ T|_{S_2 \cup C}$, which is continuous on $S_2 \cup C$, we have $g(z) \to g(z_0)$ as $z \to z_0$, $z \in S_2$.

For $z_0 \in C$, since both of $\displaystyle\lim_{z \to z_0,\ z \in S_2} g(z)$ and $\displaystyle\lim_{z \to z_0,\ z \in S_2} g(z)$ are equal to $g(z_0)$, we have the continuity of g on C.

Therefore g is a continuous function defined on S.

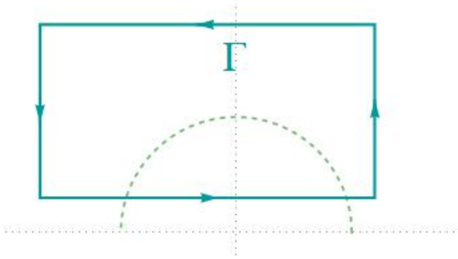
接下來證明沿 S 中的長方形路徑積分都是 0

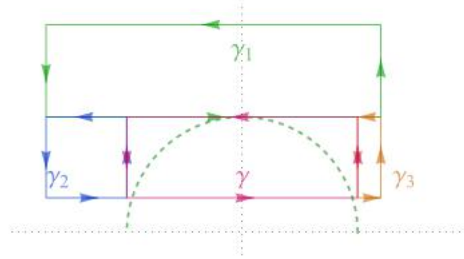
For a rectangle path Γ in S. Since g is analytic on S_1 and on S_2, and continuous on S, the integral $\int_{\partial R} g(z)\,dz = 0$ for every rectangle R with $R \subset S_1 \cup C$ or $R \subset S_2 \cup C$.

Therefore we way assume Γ lies in $\{\ x + iy \mid (x, y) \in [-1, 1], (0, 1]\ \}$, otherwise we may write $\int_\Gamma g(z)\,dz$ as

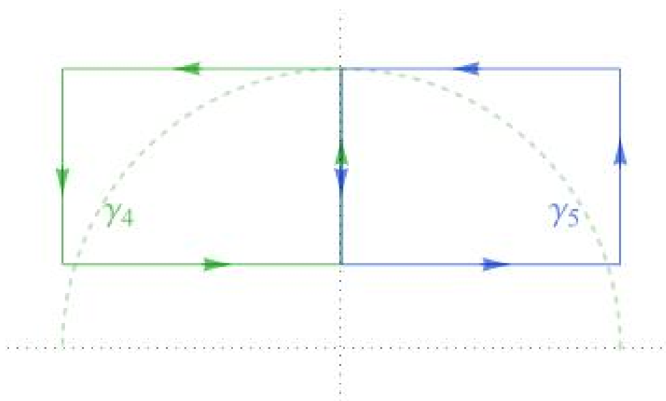
$$\int_\Gamma g(z)\,dz = \int_\gamma g(z)\,dz + \int_{\gamma_1} g(z)\,dz + \int_{\gamma_2} g(z)\,dz + \int_{\gamma_3} g(z)\,dz = \int_\gamma g(z)\,dz,$$

where γ, γ_1, γ_2, and γ_3 are as the following figure.





We may also decompose $\int_\gamma$ into two parts as the following figure, then we estimate each part.



Let $\Gamma = \partial R$ be a rectangular path, where $R = \{\ x + iy \mid (x, y) \in [a, b] \times [c, d]\} \subset [-1, 0] \times (0, 1]$.

For a partition $\{\ x_0, \ldots, x_n\ \}$ with $a = x_0 < x_1 < \cdots < x_n = b$ of $[a, b]$, let $y_j = \sqrt{1 - x_j^2}$ for each j, and let $R_{k,j} = \{\ x + iy \mid (x, y) \in [x_{k-1}, x_k] \times [y_{j-1}, y_j]\ \}$.

Then $R_{k,j} \subset S_1 \cup C$ for $k > j$ and $R_{k,j} \subset S_2 \cup C$ for $k < j$.

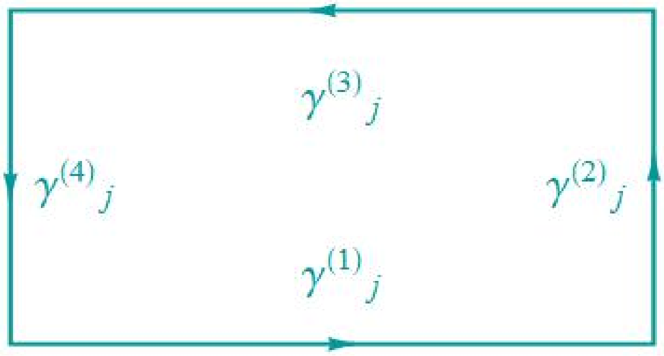
$\Rightarrow \int_\Gamma g(z)\,dz = \sum_{j=1}^n \int_{\partial R_{j,j}} g(z)\,dz.$

Since g is uniformly continuous on R, for every $\epsilon > 0$, there are $\delta > 0$ such that

$$|g(z) - g(w)| < \epsilon \qquad \forall\ z, w \in R \quad \text{with} \quad |z - w| < \delta.$$

Let the partition $\{\ x_0, \ldots, x_n\ \}$ of $[a, b]$ be chosen so that $|z - w| < \delta$ for every $z, w \in R_{j,j}$ for each j.

Let $\partial R_{j,j} = \gamma_j^{(1)} + \gamma_j^{(2)}, \gamma_j^{(3)}, \gamma_j^{(4)}$, where $\gamma_j^{(k)}$ are line segments as the following figure:



Then

$$\begin{aligned}\left|\int_{\gamma_j^{(1)}} g(z)\,dz + \int_{\gamma_j^{(3)}} g(z)\,dz\right| &= \left|\int_{x_{j-1}}^{x_j} g(x+iy_{j-1}) - g(x+iy_j)\,dx\right| \\ &\le \int_{x_{j-1}}^{x_j} |g(x+iy_{j-1}) - g(x+iy_j)|\,dx \\ &< \int_{x_{j-1}}^{x_j} \epsilon\,dx = \epsilon(x_j - x_{j-1})\end{aligned}$$

for each j.

Similarly, we have

$$\left|\int_{\gamma_j^{(1)}} g(z)\,dz + \int_{\gamma_j^{(3)}} g(z)\,dz\right| < \epsilon(y_j - y_{j-1})$$

for each j.

$\Rightarrow \left|\int_\Gamma g(z)\,dz\right| < \epsilon(b-a) + \epsilon(d-c) < 2\epsilon.$

Since $\epsilon > 0$ is arbitrary, the integral is 0.

Similarly for integrals of g on rectangular paths in $[0,1] \times (0,1]$, therefore the integral of g on every rectangular part in S is 0.

By Morera's theorem, g is analytic on S. □