Complex Analysis — Homework 5

1. Find a power series expansion for e^z about any point $a \in \mathbb{C}$.
2. Find a power series expansion for $1/z$ around $z = 1 + i$.
3. f is called an **odd** function if $f(-z) = -f(z)$ for all z; f is called **even** if $f(z) = f(-z)$ for all z.
   (a) Show that an odd entire function has only odd terms in its power series expansion about $z = 0$.
   (b) Show that an even entire function has only even terms in its power series expansion about $z = 0$.
4. Show that
$$f^{(k)}(a) = \frac{k!}{2\pi i} \int_C \frac{f(\omega)}{(\omega - a)^{k+1}}\, d\omega, \quad k = 1, 2, \cdots$$
   where C is a circle surrounding the point a and f is entire.
5. (a) Suppose an entire function f is bounded by M along $|z| = R$. Show that the coefficients C_k in its power series expansion about 0 satisfy
$$|C_k| \leq \frac{M}{R^k}.$$
   (b) Suppose a polynomial is bounded by 1 in the unit disc. Show that all its coefficients are bounded by 1.
   (c) (An alternative proof of Liouville's Theorem) Suppose that $|f(z)| \leq A + B|z|^k$ and that f is entire. Show that all the coefficients C_j, $j > k$, in its power series expansion are 0.
6. Prove that a nonconstant entire function *cannot* satisfy the two equations
   i. $f(z + 1) = f(z)$
   ii. $f(z + i) = f(z)$

   for all z.
7. A **real polynomial** is a polynomial whose coefficients are all real. Show that every real polynomial is equal to a product of real linear and quadratic polynomials.
8. Suppose P is a polynomial (with complex coefficients) such that $P(z)$ is real if and only if z is real. Prove that P is linear.
9. Show that if f is analytic in $|z| < 2$, there must be some positive integer n such that $f(1/n) \neq 1/(n+1)$.