Complex Analysis — Homework 4

1. Suppose F is analytic on an open connected set $U \subset \mathbb{C}$. Prove that if $F'(z) = 0$ for any $z \in U$, then F is a constant.

2. Show that, if f is a continuous *real-valued* function and $|f(z)| \leq 1$, for $|z| = 1$, then
$$\Big| \int_C f(z)\, dz \Big| \leq 4,$$
where $C = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle.

3. Let C be the curved given by $z(t) = e^{it}$, $0 \leq t \leq 2\pi$. Calculate

   (a) $\displaystyle\int_C z^k\, dz$ for any integer k;

   (b) $\displaystyle\int_C \sin z\, dz$;

   (c) $\displaystyle\int_C (e^z)^3\, dz$;

   (d) $\displaystyle\int_C \frac{z^2}{z+3}\, dz$.

4. Evaluate $\displaystyle\int_C (z - i)\, dz$ where C is the parabolic segment:
$$z(t) = t + it^2, \quad -1 \leq t \leq 1$$

   (a) by applying Proposition 4.12,

   (b) by integrating along the straight line from $-1 + i$ to $1 + i$ and applying the Closed Curve Theorem.

5. Calculate

   (a) $\displaystyle\int_0^{1+i} z^2\, dz$;

   (b) $\displaystyle\int_0^{\pi+2i} \cos(z/2)\, dz$;

   (c) $\displaystyle\int_{2-i}^{2+i} (z-2)^3\, dz$.