# Complex Analysis — Homework 3

1. Suppose $\sum a_n z^n$ and $\sum b_n z^n$ have radii of convergence $R_1$ and $R_2$, respectively. Disprove that the radius of convergence of $\sum (a_n + b_n) z^n$ is smaller than or equal to $\min\{R_1, R_2\}$.

2. Suppose $\sum a_n z^n$ and $\sum b_n z^n$ have radii of convergence $R_1$ and $R_2$, respectively. Show that the power series

$$\sum_{n=0}^{\infty} \Big( \sum_{k=0}^{n} a_k b_{n-k} \Big) \cdot z^n$$

converges to $\Big( \sum_{n=0}^{\infty} a_n z^n \Big) \cdot \Big( \sum_{n=0}^{\infty} b_n z^n \Big)$ for $|z| < \min\{R_1, R_2\}$.

3. Let

$$f(z) = 1 + z + \frac{z^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Show that

(a) the radius of convergence of $f(z)$ is $\infty$;

(b) $f(z_1 + z_2) = f(z_1) \cdot f(z_2)$ for any $z_1, z_2 \in \mathbb{C}$;

(c) $f(z) = e^z$ for any $z \in \mathbb{C}$.

(Hint: Problem 5 in Homework 2.)

4. Show that, for any $z \in \mathbb{C}$,

(a) $\sin z = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^n,$

(b) $\cos z = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^n.$

5. Show that there is no power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ such that

(i) $f(z) = 1$ for $z = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots,$

(ii) $f'(0) > 0$.

6. Assume $\limsup_{n\to\infty} |c_n|^{1/n} < \infty$. Show that if we set

$$f(z) = \sum_{n=0}^{\infty} c_n (z - \alpha)^n,$$

then

$$c_n = \frac{f^{(n)}(\alpha)}{n!}.$$

7. Let $C_1$ be the curved given by $\gamma(t) = \cos(\pi t) + i \sin(\pi t)$, $0 \le t \le 1$, and $C_2$ be the curved given by $\sigma(t) = \cos(\pi t) - i \sin(\pi t)$, $0 \le t \le 1$. Calculate

(a) $\displaystyle\int_{C_1} z^3\, dz$ (c) $\displaystyle\int_{C_1} \frac{1}{z}\, dz$ (e) $\displaystyle\int_{C_1} \frac{1}{z^2}\, dz$

(b) $\displaystyle\int_{C_2} z^3\, dz$ (d) $\displaystyle\int_{C_2} \frac{1}{z}\, dz$ (f) $\displaystyle\int_{C_2} \frac{1}{z^2}\, dz$

8. Let $C$ be a piecewise $C^1$ curve, $f$ and $g$ be continuous functions on $C$, and $\alpha$ be any complex number. Then

$$\int_C \big(\alpha f(z) + g(z)\big)\, dz = \alpha \int_C f(z)\, dz + \int_C g(z)\, dz.$$