Complex Analysis — Homework 2

1. (a) Show that $f(z) = x^2 + iy^2$ is differentiable at all points on the line $y = x$.
   (b) Show that it is nowhere analytic.

2. Show that a nonconstant analytic function cannot map a region (i.e. open connected subset in $\mathbb{C}$) into a straight line.

3. Find all analytic functions $f = u + iv$ with $u(x, y) = x^2 - y^2$.

4. Suppose that f is an entire function of the form

$$f(x + iy) = u(x) + iv(y),$$

   where $u, v : \mathbb{R} \to \mathbb{R}$. Show that f is a linear polynomial.

5. Prove that $f(z) = e^z$, $z \in \mathbb{C}$, is the only entire function which satisfies

$$\begin{cases} f(z_1 + z_2) = f(z_1)f(z_2), & \forall\, z_1, z_2 \in \mathbb{C} \\ f(x) = e^x, & \forall\, x \in \mathbb{R}. \end{cases}$$

6. Let $z, z_1, z_2 \in \mathbb{C}$, $x, y \in \mathbb{R}$. Verify the identities

   (a) $\sin^2 z + \cos^2 z = 1$,
   (b) $(\sin z)' = \cos z$,
   (c) $(\cos z)' = -\sin z$,
   (d) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$,
   (e) $\cos(z_1 - z_2) = \cos z_1 \cos z_2 + \sin z_1 \sin z_2$,
   (f) $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$.

7. Let

$$f(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n$$

   be a polynomial with complex coefficients. Prove that
   (a) there exist polynomials $p(x, y), q(x, y) \in \mathbb{R}[x, y]$ with real coefficients such that

$$f(x + iy) = p(x, y) + iq(x, y), \qquad \forall\, x, y \in \mathbb{R};$$

   (b) the polynomials $p(x, y), q(x, y)$ satisfy the Cauchy–Riemann equation

$$\begin{cases} p_x = q_y, \\ p_y = -q_x. \end{cases}$$

8. Let $p(x, y), q(x, y) \in \mathbb{R}[x, y]$ be polynomials with real coefficients, and let $f : \mathbb{C} \to \mathbb{C}$ be the function

$$f(x + iy) = p(x, y) + iq(x, y), \qquad \forall\, x, y \in \mathbb{R}.$$

   Prove that there exist $\alpha_0, \cdots, \alpha_n \in \mathbb{C}$ such that

$$f(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n, \qquad \forall\, z \in \mathbb{C}$$

   if and only if $p(x, y), q(x, y)$ satisfy the Cauchy–Riemann equation

$$\begin{cases} p_x = q_y, \\ p_y = -q_x. \end{cases}$$

   (Hint: Read the proof of Proposition 2.3.)

9. Find the radius of convergence of

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$,
   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!} z^n$.