Complex Analysis — Homework 1

1. Prove that the complex numbers form a field.
2. Read Section 1.4 and review the relevant concepts.
3. Express in the form $a + bi$:

   (a) $\dfrac{1}{6+2i}$ (b) $\dfrac{(2+i)(3+2i)}{1-i}$ (c) $\left(-\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right)^4$

4. Let $z, z_1, z_2 \in \mathbb{C}$. Prove the following:
   (a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.
   (b) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$.
   (c) $\overline{P(z)} = P(\overline{z})$, for any polynomial P with real coefficients.
   (d) $\overline{\overline{z}} = z$.
   (e) $\overline{z^n} = \overline{z}^n$.
   (f) $|z|^2 = \overline{z} \cdot z$.
   (g) $\max\{|\operatorname{Re} z|, |\operatorname{Im} z|\} \leq |z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|$. When do the equalities hold?
5. Suppose P is a polynomial with real coefficients. Show that $P(z) = 0$ if and only if $P(\overline{z}) = 0$.
6. Let $\Sigma = \{(\xi, \eta, \zeta) \in \mathbb{R}^3 : \xi^2 + \eta^2 + (\zeta - \frac{1}{2})^2 = \frac{1}{4}\}$, and let $\pi : \Sigma \setminus \{(0,0,1)\} \to \mathbb{C}$ be the stereographic projection. Suppose $S \subset \Sigma$. Prove or disprove
   (a) S is a circle which contains $(0,0,1)$ if and only if $\pi\big(S \setminus \{(0,0,1)\}\big)$ is a line in $\mathbb{C}$.
   (b) S is a circle which doesn't contain $(0,0,1)$ if and only if $\pi(S)$ is a circle in $\mathbb{C}$.
7. Prove Proposition 2.5.
8. Prove that the composition of complex differentiable functions is complex differentiable.
9. Suppose f is analytic in a **region** (i.e. open connected set) U and $f' \equiv 0$ in U. Show that f is constant.
10. Show that there are no analytic functions $f = u + iv$ with $u(x,y) = x^2 + y^2$.