Complex Analysis — Homework 10

1. Let N be a positive integer, C_N be the union of $\{z \in \mathbb{C} : \operatorname{Re} z = \pm(N+\frac{1}{2}), -(N+\frac{1}{2}) \le \operatorname{Im} z \le (N+\frac{1}{2})\}$ and $\{z \in \mathbb{C} : \operatorname{Im} z = \pm(N+\frac{1}{2}), -(N+\frac{1}{2}) \le \operatorname{Re} z \le (N+\frac{1}{2})\}$. Show that

$$|\cot \pi z| < 2, \qquad \forall\, z \in C_N.$$

2. Evaluate
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$,
   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$,
   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$.

3. Show that

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}$$

as long as $|x| < \dfrac{1}{4}$.

4. Let $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$. Show that
   (a) f is *not* locally 1-1 at 0;
   (b) f is *not* 1-1 in $\mathbb{C} - \{0\}$;
   (c) f is locally 1-1 in $\mathbb{C} - \{0\}$.

5. Let R be a simply connected domain which is not equal to $\mathbb{C}$. Show that there exists no conformal mapping of $\mathbb{C}$ onto R.

6. Find the image of the circle $|z| = 1$ under the mappings
   (a) $\omega = \dfrac{1}{z}$,
   (b) $\omega = \dfrac{1}{z-1}$,
   (c) $\omega = \dfrac{1}{z-2}$.

7. Let f be a bijective conformal mapping from the unit disc to the unit disc. Use Schwarz Lemma to show that if $f(0) = 0$ and $f'(0) > 0$, then f is the identity map $f(z) \equiv z$.

8. Let

$$f(z) = \frac{az+b}{cz+d}, \qquad g(z) = \frac{sz+t}{uz+v},$$

with $ad - bc \neq 0$, $sv - tu \neq 0$.
   (a) Compute $f(g(z))$ for $z \neq -d/c$.
   (b) Find s, t, u, v such that $f(g(z)) = z$ and $g(f(z)) = z$.
   (c) Show that f is a conformal mapping from $\mathbb{C} - \{-d/c\}$ onto $\mathbb{C} - \{a/c\}$.
   (d) Show that

$$\left\{ \frac{az+b}{cz+d} : ad - bc \neq 0 \right\}$$

is a group under composition.