National Tsing Hua University
Complex Analysis – Exam 4

Instructor: Hsuan-Yi Liao

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Name: ______________________________

Student ID: ______________________________

- This exam contains 10 pages (including this cover page) and 9 questions.
- Total of points is 25.
- Write down your computation or arguments in details unless otherwise stated.
- In this exam, assume
  - $\mathfrak{s}$ = your student ID;
  - $\tilde{\mathfrak{s}} = 2 + \Big|(\text{the last digit of your student ID}) - 5\Big|$.

  For example, if your student ID is 66666, then $\mathfrak{s} = 66666$ and $\tilde{\mathfrak{s}} = 3$.
- Plug numbers into the parameters $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$ in your answers.

Distribution of Marks

Question	Points	Score
1	3	
2	3	
3	3	
4	3	
5	3	
6	3	
7	3	
8	2	
9	2	
Total:	25	

1. (3 points) Show that there are no analytic functions $f = u + iv$ with $u(x, y) = x^2 + y^2$.

2. (3 points) Prove that a nonconstant entire function *cannot* satisfy the two equations

   i. $f(z+1) = f(z)$

   ii. $f(z+i) = f(z)$

   for all z.

3. Find the Laurent expansion for

   (a) (1 point) $\dfrac{1}{z^4 + z^2}$ about $z = 0$

   (b) (2 points) $\dfrac{1}{z^2 - 4}$ about $z = 2$.

4. (3 points) Find the number of zeros (counting multiplicities) of $f(z) = z^4 - 5z + 1$ in $1 \leq |z| \leq 2$

5. (3 points) Evaluate the integral $\displaystyle\int_0^\infty \frac{1}{\sqrt[3]{x}(1+x)}\,dx$.

6. (3 points) Show that

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}$$

as long as $|x| < \frac{1}{4}$.

7. (3 points) Let R be a simply connected domain and assume $z_1, z_2 \in R$. Show that there exists a conformal mapping of R onto itself, taking z_1 to z_2. (Consider two cases: $R \neq \mathbb{C}$ and $R = \mathbb{C}$.)

8. (2 points) Evaluate the integral

$$\int_0^{2\pi} \frac{1}{1 + \tilde{\mathfrak{s}} + \tilde{\mathfrak{s}} \sin x} \, dx.$$

9. (2 points) Let $R \subset \mathbb{C}$ be the open set

$$R = \{z \in \mathbb{C} : |z - 1| < 1 \text{ and } |z - i| < 1\}.$$

Find a conformal mapping f from R onto the unit disk U with the property

$$f(\frac{1}{\boldsymbol{\mathfrak{s}}} + \frac{i}{\boldsymbol{\mathfrak{s}}}) = 0.$$