Global Complex Roots and Poles Finding Algorithm Based on Phase Analysis for Propagation and Radiation Problems

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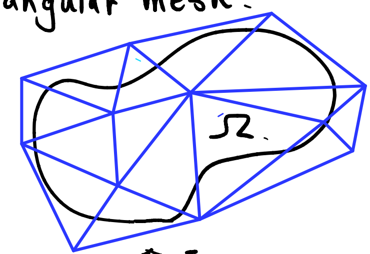
0 motivation :

In numerical analysis course, we have learned about some root finding problem for real function, for example, newton's method, bisection .... In this report, I read a paper about the root finding algorithm of the complex function based on Phase analysis and some application.

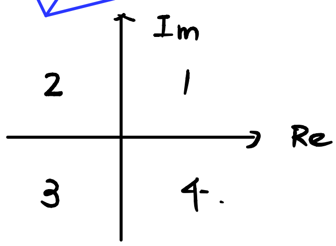
1. algorithm – step A

- $f(z)$: an analytic function.
  $\Omega \subseteq \mathbb{C}$. to find zeros and poles of f in Ω
- step A : preliminary estimation.
  - substep A1 : mesh.
    we first cover Ω with a regular triangular mesh of N nodes and P edges.



  - substep A2 : evaluation.
    for each node z_n, compute $f_n = f(z_n)$.
    and consider the function Q_n.

$$Q_n = \begin{cases} 1, & 0 \le \arg f(z_n) < \pi/2 \\ 2, & \pi/2 \le \arg f(z_n) < \pi \\ 3, & \pi \le \arg f(z_n) < 3\pi/2 \\ 4, & 3\pi/2 \le \arg f(z_n) < 2\pi \end{cases}$$

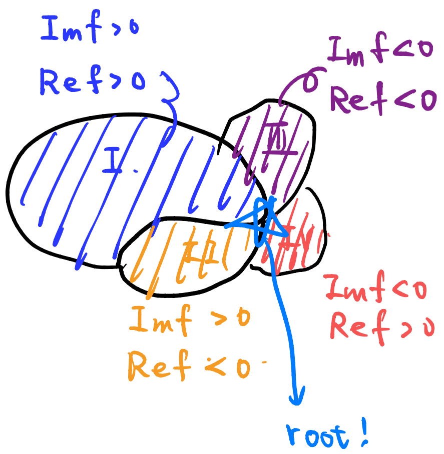


$\Rightarrow$ after this step, every node z_n has an associate $Q_n \in \{1, 2, 3, 4\}$.

  - substep A3 : candidate select.
    each edge E has to endpoint E_1, E_2.
    we can compute $\Delta Q_E = Q_{E_1} - Q_{E_2}$. $\Delta Q_E \in \{\pm 3, \pm 2, \pm 1, 0\}$.

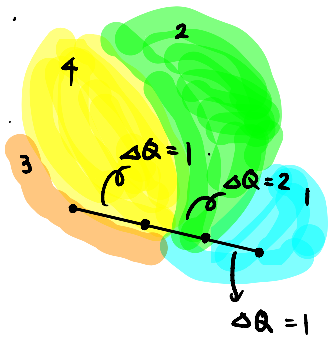
Important fact 1

the pole and root of function f must located at the point where 4 different quadrants meet



Important fact 2:

if 4 nodes accross 4 different quadrants. then $\exists\, p$ with endpoints in the 4 nodes s.t. $|\Delta Q_p| = 2$.

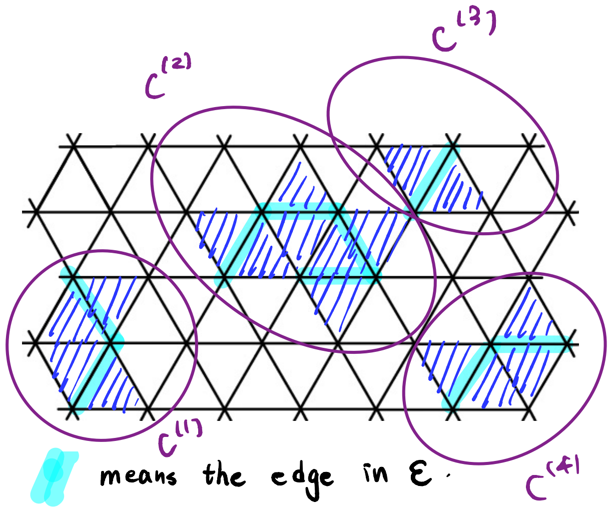


$\Rightarrow$ all edge with $|\Delta Q_p| = 2$ must have protential to become root or pole $\Rightarrow$ collect it.

$\mathcal{E} = \{ p \mid |\Delta Q_p| = 2 \}$.

- **substep A4:** candidate region.

  we collect the triangle which at least one edge is in $\mathcal{E}$. in C. C can divided into $C^{(k)}$, $C^{(k)}$ creates close contour surrounding region.



means the edge in $\mathcal{E}$.

/// means the triangle in C.

- **substep A5:** verification.

  here applies the thm learned in class.

-thm: let f be a meromorphic function in a simply connected domain D. with poles $p_1 \dots p_m$ and zeros $z_1 \dots z_n$. if γ is a closed piecewise C^1 curve in D not passing p_i, z_j. then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n} \operatorname{ord}(z_k)\cdot n(\gamma, z_k) - \sum_{j=1}^{m} \operatorname{ord}(p_j)\cdot n(\gamma, p_j)$$

let $q = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz$. If only one candidate point is surrounding by γ. then q can be

1) positive integer $\rightarrow$ root of order q.

2). negative integer $\rightarrow$ pole of order $-q$.

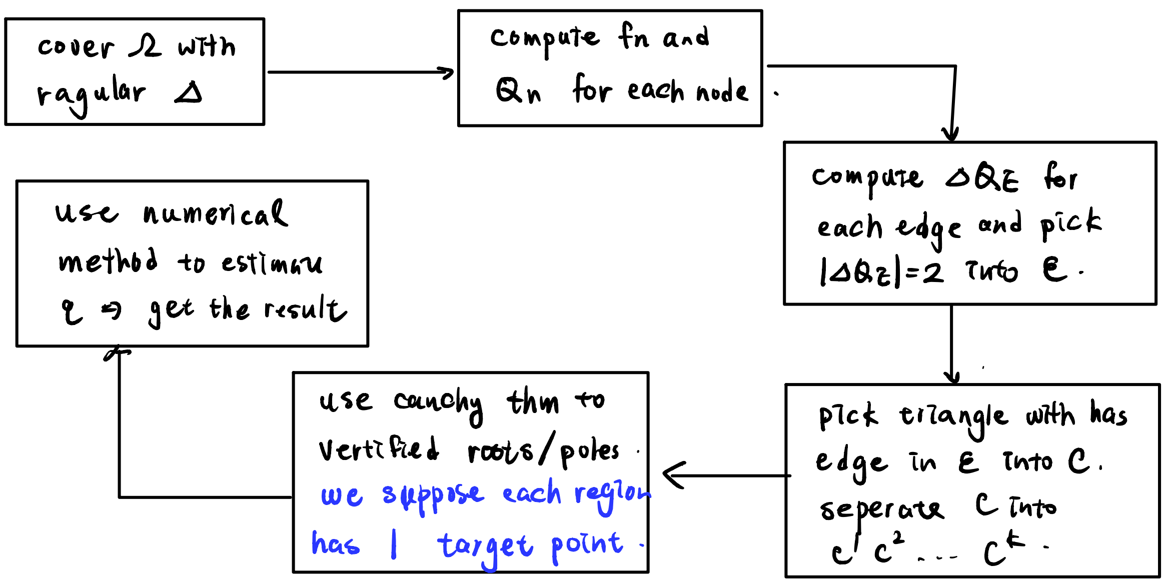
3). $0 \longrightarrow$ regular point.

$\Rightarrow$ there is no need to calculate the integral, we just want to know the sign of q; we approximate it by sampling $z_1 \dots z_p$, $z_{p+1} = z_1$ on γ.

$\Rightarrow$ $q = \frac{1}{2\pi} \sum_{i=1}^{p} \arg \frac{f(z_{i+1})}{f(z_i)}$. $z_1 \dots z_{p+1}$ must satisfied

$\left| (\arg f(z))|_{z \in (z_p, z_{p+1})} \right| \le \pi$.

2. summary and diagram of algorithm step A



- question:

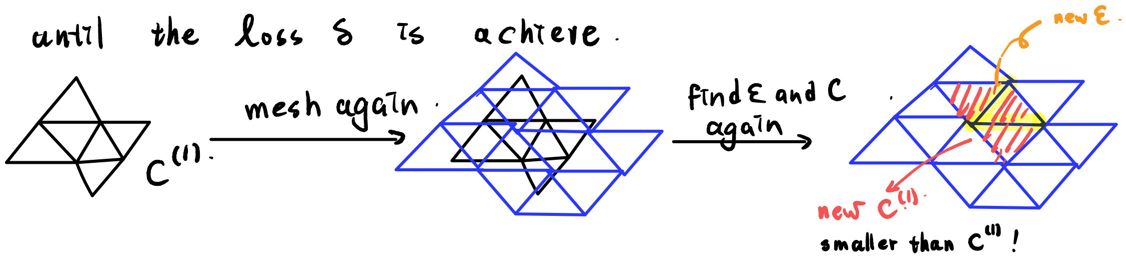
(1). there might be more than one target point in $C^{(k)}$.

(2). we only know the type of point in $C^{(k)}$,

$\boxed{\Rightarrow \text{need to improve the accuracy!}}$

3. algorithm: step B: mesh refinement.

to improve the accuracy. we can mesh the region $C^{(k)}$ until the loss δ is achieve.



- in step A1, we also can pick little enough mess step Δr to achieve better accuracy.

4. Limitation and effectiveness

- If the discretization of the region is proper, no root or pole will be missed!
- but in practice, Δr can be even greater than the distance of two target point.
- a modification is to rise the moment of the integral to lower the risk of missing roots/poles.

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)} z^m dz = \sum_{k\in\{roots\}} \left(z^{(k)}\right)^m - \sum_{k\in\{poles\}} \left(z^{(k)}\right)^m .$$

5. summary.

In complex analysis classes, starts from taylor series, some numerical method of finding roots use polynomial to approximate f. But $f(z) = e^z = \sum_{i=1}^{\infty} \frac{z^i}{i!}$ has no roots, but if use $\sum_{i=1}^{N} \frac{z^i}{i!}$ to approximate e^z, it has root = 0. next the class develop some line integral property and some facts about closed curve, and at week 11, we learned about the three types of singularities, and some method to classify it. At week 13. we study the thm which connects the line integral and poles, zeros, after all I apply those knowledge to study this paper about numerical method about complex function.

This is the last must-take class in my university study. during the 3 years of study in dep. of math, I learned lots of knowledge and have interested in statistical learning, numerical computing and convex optimization, the complex analysis course helps me to extend my interest into another aspect, it is a very useful course in my undergraduate study in math!

6- reference .

Global Complex Roots and Poles Finding Algorithm Based on Phase Analysis for Propagation and Radiation Problems by Piotr Kowalczyk