Cellular homology

Lemma 2.34 If X is a CW complex, then

(a) $H_k(X^n, X^{n-1})$ is zero for $k \neq n$ and is free abelian for $k = n$, with a basis in 1-1 correspondence with the n-cells of X

(b) $H_k(X^n) = 0$ for $k > n$. In particular, if X is finite-dimensional, then $H_k(X) = 0$ for $k > \dim X$.

(c) The map $H_k(X^n) \longrightarrow H_k(X)$ induced by the inclusion $X^n \hookrightarrow X$ is an isomorphism for $k < n$ and surjective for $k = n$.

pf

(a) is true because (X^n, X^{n-1}) is a good pair and $X^n/X^{n-1} \cong S^n \vee \cdots \vee S^n$

(b) We have the long exact seq. for (X^n, X^{n-1}):

(*) $$\cdots \to \underbrace{H_{k+1}(X^n, X^{n-1})}_{=0 \text{ if } k \neq n-1} \to H_k(X^{n-1}) \to H_k(X^n) \to \underbrace{H_k(X^n, X^{n-1})}_{=0 \text{ if } k \neq n} \to \cdots$$

if $k>n$, $\Rightarrow H_k(X^n) \cong H_k(X^{n-1}) \cong \cdots \cong H_k(X^0) = 0$ ($\because k > 0$) #

(c) finite-dim case ($X = X^N$, $\exists N$): By (*), the map induced by inclusion $X^{n-1} \hookrightarrow X^n$

$$H_k(X^n) \longrightarrow H_k(X^{n+1}) \text{ is } \begin{cases} \text{1-1 if } k \neq n \\ \text{onto if } k \neq n+1 \end{cases}$$

So if $k < n$, then $H_k(X^n) \xrightarrow{\cong} H_k(X^{n+1}) \xrightarrow{\cong} \cdots \xrightarrow{\cong} H_k(X = X^N)$ is an iso

if $k = n$, then $H_k(X^n) \xrightarrow{\text{onto}} H_k(X^{n+1}) \xrightarrow{\cong} \cdots \xrightarrow{\cong} H_k(X = X^N)$ is onto.

infinite-dim case: exer. (p.138-139) #

Let X be a CW complex. By Lemma 2.34, $H_n(X^n, X^{n-1}) \cong \mathbb{Z}^{\sigma_n}$ where σ_n = # of n-cells

Define

$$\begin{tikzcd}[column sep=small]
 & & & H_{n-1}(X^n) & & & H_{n-2}(X^{n-2}) \\
d_n := j_{n-1} \circ \partial : & \underset{\mathbb{Z}^{\sigma_n}}{\underset{\cong}{H_n(X^n, X^{n-1})}} \arrow[rr, "\partial"] & & H_{n-1}(X^{n-1}) \arrow[rr, "j_{n-1}"] \arrow[u, dashed] & & \underset{\mathbb{Z}^{\sigma_{n-1}}}{\underset{\cong}{H_{n-1}(X^{n-1}, X^{n-2})}} \arrow[ur, dashed] & \\
 & H_n(X^n) \arrow[u, dashed] & & H_{n-1}(X^{n-2}) \arrow[u, dashed] & & &
\end{tikzcd}$$

Lemma

$$d_n \circ d_{n+1} = 0$$

pf $d_n \circ d_{n+1} = H_{n+1}(X^{n+1}, X^n) \xrightarrow{\partial} H_n(X^n) \underbrace{\boxed{\xrightarrow{j} H_n(X^n, X^{n-1}) \xrightarrow{\partial}}}_{=0} H_{n-1}(X^{n-1}) \xrightarrow{j} H_{n-1}(X^{n-1}, X^{n-2})$ #

The complex $(H_n(X^n, X^{n-1}), d_n)$ is called the cellular chain complex, and the corresponding homology group is called the cellular homology of X, temporarily denoted by $H_n^{CW}(X)$

Thm 2.35

$$H_n^{CW}(X) \cong H_n(X)$$

pf

$$0 = H_n(X^{n-1}) \longrightarrow H_n(X^n) \xrightarrow[\text{onto}]{i} \boxed{H_n(X^{n+1})} \xrightarrow{j} H_n(X^{n+1}, X^n) = 0$$

$H_n(X) \cong H_n(X^{n+1})$

$\partial: H_{n+1}(X^{n+1}, X^n) \to H_n(X^n)$, $\quad j: H_n(X^n) \to H_n(X^n, X^{n-1})$ ← 1-1

$$\cdots \longrightarrow H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \longrightarrow \cdots$$

$0 = H_{n+1}(X^n) \longrightarrow H_{n+1}(X^{n+1}) \xrightarrow{j} H_{n+1}(X^{n+1}, X^n)$ ← 1-1

$0 = H_{n-1}(X^{n-2}) \longrightarrow H_{n-1}(X^{n-1}) \xrightarrow{j} H_{n-1}(X^{n-1}, X^{n-2})$ ← 1-1, $\quad \partial: H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1})$

Since $H_n(X^n) \xrightarrow{i} H_n(X^{n+1}) = H_n(X)$ is onto, we have

$$H_n(X) \cong H_n(X^n)/\ker i = H_n(X^n)/\partial(H_{n+1}(X^{n+1}, X^n)) \cong \underline{j(H_n(X^n))} / \boxed{j\partial}(H_{n+1}(X^{n+1}, X^n))$$

$\ker \partial = \ker(j\partial)$, $\quad j\partial = d_{n+1}$

$$= \ker(d_n)/\operatorname{im}(d_{n+1}) = H_n^{CW}(X) \quad \#$$

Example

Recall from Example 0.6 that

$$\mathbb{CP}^n \cong e^0 \cup e^2 \cup \cdots \cup e^{2n}$$

$\Rightarrow$ the cellular complex of $\mathbb{CP}^n$ is

$$\cdots \longrightarrow \overset{2n}{\mathbb{Z}} \longrightarrow \overset{2n-1}{0} \longrightarrow \overset{2n-2}{\mathbb{Z}} \longrightarrow \overset{2n-3}{0} \longrightarrow \cdots \longrightarrow \overset{1}{0} \longrightarrow \overset{0}{\mathbb{Z}}$$

$$\Rightarrow H_k(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z}, & k = 0, 2, 4, \cdots, 2n \\ 0, & \text{otherwise} \end{cases} \quad \#$$

Remark

Let X be a CW complex. Then

① $H_n(X) = 0$ if X has no n-cells.

② $H_n(X)$ can be generated by k elements if X has k n-cells.

③ $H_n(X) \cong \mathbb{Z}^k$ if X has k n-cells, NO $(n-1)$-cells, NO $(n+1)$-cells. (e.g. $\mathbb{CP}^n$)

Formula of $d_n: H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}, X^{n-2})$:

Def (degree)

Let $f: S^n \longrightarrow S^n$, $n > 0$. Since $H_n(S^n) \cong \mathbb{Z}$, the induced map f_* is of the form

$$f_*: \underset{\cong\mathbb{Z}}{H_n(S^n)} \longrightarrow \underset{\cong\mathbb{Z}}{H_n(S^n)}, \quad f_*(\alpha) = d\alpha.$$

The number d is called the degree of f

Properties of degree (p.134): Let $f, g : S^n \to S^n$

(a) $\deg(\mathrm{id}_{S^n}) = 1$

(b) If $f : S^n \to S^n$ is NOT onto, then $\exists\, x_0 \in S^n - f(S^n)$

$\Rightarrow f = S^n \xrightarrow{f} S^n \setminus \{x_0\} \hookrightarrow S^n$

$\Rightarrow f_* = H_n(S^n) \longrightarrow \underbrace{H_n(S^n \setminus \{x_0\})}_{=0} \longrightarrow H_n(S^n) = 0$

$\Rightarrow \deg(f) = 0$

(c) $f \simeq g \iff \deg(f) = \deg(g)$ "$\Rightarrow$" is easy "$\Leftarrow$" is proved in Cor 4.25

(d) $\deg(f \circ g) = \deg(f)\deg(g)$ exer: derive other properties on p.134

Cellular Boundary Formula

identify cells with generators

The boundary map $d_n : \underset{\cong\, \mathbb{Z}\{e_\alpha^n\}}{H_n(X^n, X^{n-1})} \longrightarrow \underset{\cong\, \mathbb{Z}\{e_\beta^{n-1}\}}{H_{n-1}(X^{n-1}, X^{n-2})}$ satisfies

$$d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta}\, e_\beta^{n-1}$$

where $d_{\alpha\beta}$ is the degree of the map

$$\partial D_\alpha^n = S_\alpha^{n-1} \xrightarrow{\text{attaching}} X^{n-1} \xrightarrow{\text{quotient}} X^{n-1}/X^{n-1} - e_\beta^{n-1} \cong S_\beta^{n-1} \qquad (n > 1)$$

$n=1$: $d_1 : H_1(X^1, X^0) \to H_0(X^0)$ If X^0 = one point, then $d_1 = 0$

$\gamma : \Delta^1 \to X$, $[\gamma] \longmapsto \gamma(1) - \gamma(0)$

$\Delta^1 \cong [0,1]$, $\gamma(0), \gamma(1) \in X^0$

Remark: $\exists$ only finite nonzero $d_{\alpha\beta}$ because S_α^{n-1} has a compact image which meets only finite e_β^{n-1}.

pf of the formula:

"e_α^n" $\longmapsto$ $d_{\alpha\beta}\, e_\beta^{n-1}$

generator $[D_\alpha^n] \in$; $\cong S_\alpha^{n-1}$; commutes ; one summand of

$$\begin{tikzcd}
H_n(D_\alpha^n, \partial D_\alpha^n) \arrow[r, "\partial", "\cong"'] \arrow[d, "\Phi_{\alpha *}"] & \tilde H_{n-1}(\partial D_\alpha^n) \arrow[r, "\Delta_{\alpha\beta *}"] \arrow[d, "\varphi_{\alpha *}"] & \tilde H_{n-1}(S_\beta^{n-1}) \\
e_\alpha^n \in H_n(X^n, X^{n-1}) \arrow[r, "\partial"] \arrow[dr, "d_n"'] & \tilde H_{n-1}(X^{n-1}) \arrow[r, "q_*"] \arrow[d, "j"] & \tilde H_{n-1}(X^{n-1}/X^{n-2}) \cong \vee_\beta S_\beta^{n-1} \arrow[u, "q_{\beta *}"'] \arrow[d, "\cong"] \\
& H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "\cong", "\leftarrow \text{Prop 2.22}"'] & H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2})
\end{tikzcd}$$

where: $D_\alpha^n \hookrightarrow X^{n-1} \sqcup_\alpha D_\alpha^n \twoheadrightarrow X^n \to X$

- Φ_α is the characteristic map of e_α^n, and φ_α is its attaching map
- $q : X^{n-1} \to X^{n-1}/X^{n-2}$ is the quotient map
- q_β:

$$\begin{tikzcd}
X^{n-1}/X^{n-2} \arrow[r, two heads] & X^{n-1}/X^{n-1} - e_\beta^{n-1} & S_\beta^{n-1} = D_\beta^{n-1}/\partial D_\beta^{n-1} \arrow[l, "\cong"', "\uparrow \text{essentially } \Phi_\beta"] \\
& X^{n-1} \arrow[u] & D_\beta^{n-1} \arrow[u] \arrow[l, "\Phi_\beta"]
\end{tikzcd}$$

...

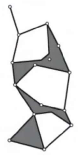
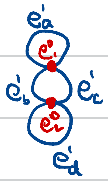
...

- $\Delta_{\alpha\beta} : \partial D^n_\alpha \to S^{n-1}_\beta = \partial D^n_\alpha \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q} X^{n-1}/X^{n-2} \xrightarrow{q_\beta} S^{n-1}_\beta$

By the commutativity of the diagram, we have

$d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta}\, e^{n-1}_\beta$ #

Example (See the board of week 5)

 homotopic to 

$X_0 = :$, $X_1 = X$

cellular cx:

$e_a' \longmapsto e^0_1 - e^0_1 = 0$ ← endpoint - startpoint

$e_d' \longmapsto e^0_2 - e^0_2 = 0$

$0 \to \mathbb{Z}^4 \xrightarrow{d_1} \mathbb{Z}^2 \to 0$

$e_b' \longmapsto e^0_1 - e^0_2$

$e_c' \longmapsto e^0_1 - e^0_2$ ← orientation NOT important here because will generate a subgp

$H_0(X) = \mathbb{Z}^2/\langle(1,-1)\rangle \cong \mathbb{Z}$

$\Rightarrow H_1(X) = \ker(d_1)$

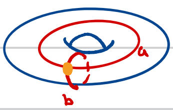
$= \langle (1,0,0,0), (0,0,0,1), (0,1,-1,0) \rangle$

$\cong \mathbb{Z}^3$

other $H_k = 0$ #

Example 2.36 (orientable closed surface)

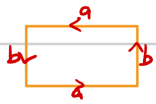
Remark: will explain later: $\deg(S^1 \to S^1 : z \mapsto z^n) = n$



$T = M_1$

$= e^0 \cup e^1_a \cup e^1_b \cup e^2$

$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{d_1} \mathbb{Z} \to 0$, $d_1 = 0$



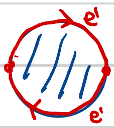
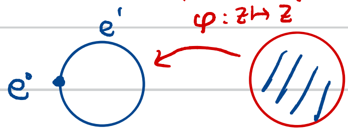
$\partial D^2 = S^1 \to X^1 =$  $\longrightarrow X^1/X^1 \setminus e_a \cong$  has deg $1-1=0$

Similarly, $\deg(S^1 \to X^1/X^1\setminus e_b \cong S^1_b) = 0 \Rightarrow d_2 = 0$

$\Rightarrow H_k(T) = \begin{cases} \mathbb{Z}, & k=0,2 \\ \mathbb{Z}^2, & k=1 \\ 0, & \text{otherwise} \end{cases}$

exer: Compute homology of orientable surface of genus g (see picture on p.5)

Example 2.37 (nonorientable closed surface)

① $N_1 \cong \mathbb{R}P^2 \cong$  (Also see wiki: Boy's surface) $\cong e^0 \cup e^1 \cup_\varphi e^2$ 

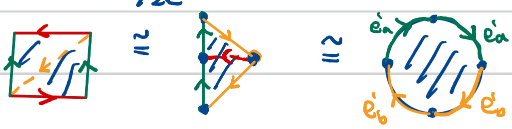
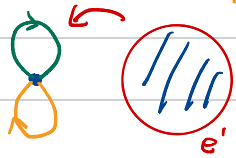
See Example 0.4

$\deg(\varphi) = 2$

Cellular complex of $\mathbb{R}P^2$:

$0 \to \underset{e^2}{\mathbb{Z}} \xrightarrow{\times 2} \underset{e^1}{\mathbb{Z}} \xrightarrow{0} \underset{e^0}{\mathbb{Z}} \to 0$

$\Rightarrow H_0(\mathbb{R}P^2) \cong \mathbb{Z}$, $H_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$, $H_2(\mathbb{R}P^2) = 0$

② $N_2 \cong$ Klein bottle $K \cong$  $\cong e^0 \cup e^1_a \cup e^1_b \cup_\varphi e^2$ 

See wiki: Klein bottle

Cellular complex of K:

$0 \to \underset{e^2}{\mathbb{Z}} \xrightarrow{1 \mapsto (2,2)} \underset{e_a'\ \ e_b'}{\mathbb{Z} \oplus \mathbb{Z}} \xrightarrow{0} \underset{e^0}{\mathbb{Z}}$

$\Rightarrow H_0(K) \cong \mathbb{Z}$, $H_1(K) \cong \mathbb{Z}^2/\langle(2,2)\rangle \cong \mathbb{Z} \oplus \mathbb{Z}_2$, $H_2(K) = 0$

$(a, b) \longleftrightarrow (a-b, b)$

$(x+y, y) \longleftrightarrow (x, y)$

exer: $H_k(N_g) = ?$

N_g = closed nonorientable surface, See Example 2.37

Further reading:

Classification of closed surfaces < orientable, genus g, $g \geq 0$ / nonorientable, genus g, $g \geq 1$

Google: classification of surfaces